SOME GEOMETRICAL ASPECTS OF HARMONIC CURVES IN A COMPLEX FINSLER SPACE

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In this note we make a short study of the geometry of curves in a complex Finsler space. For harmonic curves we obtain an equivalent characterization to that from [Ni]. A special discussion concerns the holomorphic curves.

1 Introduction

The geometry of harmonic maps knows a large references, being itself a complex subject in differential geometry. It is difficult to quote a minimal reference of this topic, we believe that [E-L] and [Gu] should give the reader some insight of the techniques and fundamental results in harmonic map theory.

In the last decades a new notion has drawed our attention for its various applications in complex geometry, namely that of complex Finsler spaces. Roughly speaking this is the geometry of the holomorphic tangent bundle, as complex manifold, equipped with a homogeneous Hermitian metric derived from a positive Lagrangian. For reference we confine here to send the reader only to some more recent comprehensive studies: [A-P, Ai, Mu, Wo].

In order to investigate the complex geodesics of a complex Finsler space, Vesentini, [Ve], studied the holomorphic isometric embending of the unit disc, equipped with the Poincaré metric, into a complex Finsler space. Later on, Abate and Patrizio, [A-P], realized a characterization of the Kobayashi metrics using the existence of such complex geodesics and proved a classification of Kähler Finsler manifolds of constant holomorphic curvature.

In this definition of complex geodesics one can recognize the more general notion of harmonic maps from a compact Riemann surface into a complex Finsler space. The investigation of harmonic maps of a complex Finsler space knows yet some recent results: [Br, Mo, Ni]. In a recent paper we investigated the geometry of a class of particular harmonic maps, namely the holomorphic curves of a complex Finsler space, [Mu2]. The holomorphy of the curve is a restrictive request but nonetheles of interest. In some weakly Kähler circumstances of positive curvature, Nishikawa ([Ni]) proved that a harmonic map (the critical points of $\bar{\partial}-$ energy) from a Riemanian sphere $P_1(\mathbf{C})$ to the complex Finsler manifold is either holomorphic or antiholomorphic.

In this note our goal is to investigate some general aspects in geometry of harmonic curves of a complex Finsler space. We rediscover a result of Nishikawa for harmonic curves using a different way. Some remarks concerning the holomorphic curves are done at the end.

2 Harmonic curves of a complex Finsler space

Let (M, F) be a complex Finsler space. In this note we are faithful to our terminology and notations from [Mu]. The complex coordinates in a local chart of M will be $(z^k)_{k=\overline{1,n}}$. Consider $\widetilde{M} \hookrightarrow M$ an embendded subspace of $\dim_C \widetilde{M} = 1$ and $f : \widetilde{M} \to M$ a smooth map, locally given by $f : w \to z^k(w)$. Generally, if f is not holomorphic then $\partial z^k / \partial \bar{w} \neq 0$.

Let $T\widetilde{M}$ and TM be the corresponding tangent spaces and $df : T\widetilde{M} \to TM$ the tangent map, which can be extended to the complexified tangent spaces, $d_C f : T_C\widetilde{M} \to T_C M$.

We have the decomposition $T_C M = T'M \oplus T''M$ into holomorphic and antiholomorphic subbundles and a similar decomposition is obtained for $T_C\widetilde{M}$. The set of tangent vectors $(z^k, \eta^k = \frac{dz^k}{dt})$ to the curves $c : t \to z^k(t)$, with $t \in R$, generate the coordinate systems of the manifold $T'M$. So, the local coordinates in a local chat of $T'M$ will be denoted by (z^k, η^k).

Analogously let (w, θ) be a system of local coordinates of $T'\widetilde{M}$ manifold, where $\theta = \frac{dw}{dt}$ is the tangent vector to the curve through w. By this construction $d_C f$ has the following local expression

$$z^k = z^k(w)\,; \quad \eta^k = \theta\frac{\partial z^k}{\partial w} + \bar{\theta}\frac{\partial z^k}{\partial \bar{w}} \tag{2.1}$$

and will be called a *complex curve* of the (M, F) Finsler space.

We note that $d_C f$ does not preserve the holomorphic tangent bundles, except for the holomorphic maps, but it has the following components: $\partial f : T'\widetilde{M} \to T'M$, $\bar{\partial} f : T''\widetilde{M} \to T'M$ and $\partial\bar{f} : T'\widetilde{M} \to T''M$, $\bar{\partial}\bar{f} : T''\widetilde{M} \to T''M$, locally given by:

$$\begin{aligned} \partial f(\frac{\partial}{\partial w}) &= \frac{\partial z^k}{\partial w}\frac{\partial}{\partial z^k}\,; \quad \bar{\partial} f(\frac{\partial}{\partial \bar{w}}) = \frac{\partial z^k}{\partial \bar{w}}\frac{\partial}{\partial z^k} \\ \partial \bar{f}(\frac{\partial}{\partial w}) &= \frac{\partial \bar{z}^k}{\partial w}\frac{\partial}{\partial \bar{z}^k}\,; \quad \bar{\partial} f(\frac{\partial}{\partial \bar{w}}) = \frac{\partial \bar{z}^k}{\partial \bar{w}}\frac{\partial}{\partial \bar{z}^k} \end{aligned} \tag{2.2}$$

Obviously, we have $\partial\bar{f} = \overline{\bar{\partial} f}$ and $\bar{\partial}\bar{f} = \overline{\partial f}$.

In the corresponding complexified tangent bundles $T_C T'\widetilde{M}$ and $T_C T'M$ the natural frames are $\{\frac{\partial}{\partial w}, \frac{\partial}{\partial \bar{w}}, \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \bar{\theta}}\}$ and $\{\frac{\partial}{\partial z^k}, \frac{\partial}{\partial \bar{z}^k}, \frac{\partial}{\partial \eta^k}, \frac{\partial}{\partial \bar{\eta}^k}\}$ respectively, which are tied by:

$$\begin{aligned} \frac{\partial}{\partial w} &= B^k\frac{\partial}{\partial z^k} + B^{\bar{k}}\frac{\partial}{\partial \bar{z}^k} + (\dot{B}^k_0 + \dot{B}^k_{\bar{0}})\frac{\partial}{\partial \eta^k} + (\dot{B}^{\bar{k}}_0 + \dot{B}^{\bar{k}}_{\bar{0}})\frac{\partial}{\partial \bar{\eta}^k} \\ \frac{\partial}{\partial \theta} &= B^k\frac{\partial}{\partial \eta^k} + B^{\bar{k}}\frac{\partial}{\partial \bar{\eta}^k}, \end{aligned} \tag{2.3}$$

where we set $B^k = \frac{\partial z^k}{\partial w}$, $B^{\bar{k}} = \frac{\partial \bar{z}^k}{\partial w}$; $\dot{B}^k_0 = \theta\frac{\partial^2 z^k}{\partial w^2}$, $\dot{B}^k_{\bar{0}} = \bar{\theta}\frac{\partial^2 z^k}{\partial w \partial \bar{w}}$ and $\dot{B}^{\bar{k}}_0 = \theta\frac{\partial^2 \bar{z}^k}{\partial w^2}$, $\dot{B}^{\bar{k}}_{\bar{0}} = \bar{\theta}\frac{\partial^2 \bar{z}^k}{\partial w \partial \bar{w}}$.

Indeed, for the conjugates we have $\frac{\partial}{\partial \bar{w}} = \overline{\frac{\partial}{\partial w}}$ and $\frac{\partial}{\partial \bar{\theta}} = \overline{\frac{\partial}{\partial \theta}}$.

By $VT'M$ we denote the vertical distribution spanned locally by $\{\frac{\partial}{\partial \eta^k}\}$. A supplementary distribution to the vertical distribution, i.e. $T'T'M = VT'M \oplus HT'M$, is a complex nonlinear connection, briefly (*c.n.c.*). A local frame in $HT'M$ will be denoted with $\{\frac{\delta}{\delta z^k} = \frac{\partial}{\partial z^k} - N^j_k\frac{\partial}{\partial \eta^j}\}$ and we say that it is adapted if its changes follow the rule ([Mu]): $N'^i_j\frac{\partial z'^j}{\partial z^k} = \frac{\partial z'^i}{\partial z^j}N^j_k + \frac{\partial^2 z'^i}{\partial z^j \partial z^k}\eta^j$.

Now let us assume that (M, F) is a complex Finsler space, with $g_{i\bar{j}} = \partial^2 F^2/\partial\eta^i\partial\bar{\eta}^j$ the Finsler metric tensor.

There exists a special Hermitian connection of (1,0)- type, namely the Chern-Finsler connection, with some remarkable properties ([A-P, Mu, Ai]): $D\Gamma(N) = (L^i_{jk}, 0, C^i_{jk}, 0)$, where

$$N^j_k = g^{\bar{m}j}\frac{\partial^2 F^2}{\partial z^k \partial \bar{\eta}^m}\,; \quad L^i_{jk} = g^{\bar{m}i}\frac{\delta g_{j\bar{m}}}{\delta z^k}\,; \quad C^i_{jk} = g^{\bar{m}i}\frac{\partial g_{j\bar{m}}}{\partial \eta^k}. \tag{2.4}$$

By the same way we can consider the vertical distribution $VT'\widetilde{M}$, locally spanned by $\{\frac{\partial}{\partial \theta}\}$, and from (2.3) we remark that it is a subdistribution of $V_C T'M$.

An immediate conclusion is that $(\widetilde{M}, \widetilde{F})$, where $\widetilde{F}$ is induced by (2.1) from $F(z, \eta)$, is not always a complex Finsler subspace, forasmuch $\widetilde{F}$ generally is not necessarily a Finsler function. A special case of interest is when f is a holomorphic curve, and this situation was considered in [Mu2]. Some results about holomorphic curves will be recalled here at the end of the paper.

The main problem in discussion now is that of first variation of a complex curve. Let us consider f restricted to a compact domain D of $\widetilde{M}$, with $\widetilde{M}$ being a compact Riemann surface. Following [Ni] the $\bar{\partial}$-energy of f is given by

$$E_{\bar{\partial}}f = \int_D F^2\left(f(w), \bar{\partial}f(\frac{\partial}{\partial\bar{w}})\right)\frac{\sqrt{-1}}{2}dw \wedge d\bar{w}.$$

In account to homogeneity of the Finsler function F, this definition is independent of the choice of the local coordinates w.

Let $f_t(w) = z^k(w) + tv^k(w)$ be a variation of f, with $f_0(w) = f(w)$. The corresponding variation of $\bar{\partial}$-energy $\frac{\partial E_{\bar{\partial}}f_t}{\partial t}\;|_{t=0}$, leads after a classical variational argument to the following type of Euler-Lagrange equations:

$$\frac{\partial F^2}{\partial z^i} - \frac{\partial}{\partial\bar{w}}\left(\frac{\partial F^2}{\partial\eta^i}\right) = 0.$$

Next, computing $\frac{\partial}{\partial\bar{w}}$ by means of z^i and η^i, according to the first formula (2.1), the Euler-Lagrange equations yield

$$(g_{i\bar{j}}\frac{\partial^2\bar{z}^j}{\partial w\partial\bar{w}} + \frac{\partial^2F^2}{\partial\bar{z}^j\partial\eta^i}\frac{d\bar{z}^j}{d\bar{w}}) + (g_{ij}\frac{\partial^2z^j}{\partial\bar{w}^2} + \frac{\partial^2F^2}{\partial z^j\partial\eta^i}\eta^j - \frac{\partial F^2}{\partial z^i}) = 0,$$

where $g_{ij} = \partial F^2/\partial\eta^i\partial\eta^j$.

Definition 2.1 *We call a harmonic map of the complex Finsler space (M, F), a solution of the system:*

$$g_{i\bar{j}}\frac{\partial^2\bar{z}^j}{\partial w\partial\bar{w}} + \frac{\partial^2F^2}{\partial\bar{z}^j\partial\eta^i}\frac{d\bar{z}^j}{d\bar{w}} = 0\;;\quad g_{ij}\frac{\partial^2z^j}{\partial\bar{w}^2} + \frac{\partial^2F^2}{\partial z^j\partial\eta^i}\eta^j - \frac{\partial F^2}{\partial z^i} = 0. \tag{2.5}$$

By conjugation in the first set of equations from (2.5), we find

$$\frac{\partial^2z^j}{\partial w\partial\bar{w}} + N^i_j\frac{\partial z^j}{\partial w} = 0 \quad \text{where} \quad N^i_j = g^{\bar{m}i}\frac{\partial^2F^2}{\partial z^j\partial\bar{\eta}^m} \tag{2.6}$$

is the Chern-Finsler (*c.n.c.*) along the points of the harmonic map.

In complete analogy to the proof of Proposition 5.1.3 from [Mu], we can check that (2.6) is just $g_{i\bar{l}}(L^i_{jk} - L^i_{kj})\eta^j\bar{\eta}^l = 0$, and that is the weakly Kähler condition along the harmonic curve. It flows that,

Theorem 2.1 *In a weakly Kähler Finsler space (M, F), the equations of a harmonic curve are given by (2.6).*

On the other hand, since the horizontal part of Chern-Finsler connection performs the condition $L^i_{jk} = \partial N^i_k/\partial\eta^j$, in account to homogeneity of N^i_k and the fact that along a harmonic curve $\eta^k = \frac{\partial z^k}{\partial\bar{w}}$, it results that $N^i_j\frac{\partial z^j}{\partial w} = L^i_{kj}\frac{\partial z^j}{\partial w}\frac{\partial z^k}{\partial\bar{w}}$. So, by plugging this into (2.6) we recover the known expression (5.9) from [Ni] of a harmonic map, this time for a complex curve of the space (M, F).

If the curve is holomorphic, then $\frac{\partial z^i}{\partial\bar{w}} = 0$ and owing the last reasoning, it follows that any holomorphic curve is harmonic. Moreover, from (2.6) we have $N^i_j\frac{\partial z^j}{\partial w} = 0$, along the points of the holomorphic curve.

Subsequently we add some supplementary comments concerning holomorphic curves.

First, we remark that for a holomorphic curve (2.3) acquires a simplified form:

$$\frac{\partial}{\partial w} = B^k \frac{\partial}{\partial z^k} + \dot{B}^k_0 \frac{\partial}{\partial \eta^k} \ ; \quad \frac{\partial}{\partial \theta} = B^k \frac{\partial}{\partial \eta^k}, \tag{2.7}$$

and $(\widetilde{M}, \widetilde{F})$, where $\widetilde{F}(w,\theta) = F(z(w), \eta(w,\theta))$, is a holomorphic subspace of (M, F).

The holomorphic tangent bundles are preserved by a holomorphic curve f, and $VT'\widetilde{M}$ is a subdistribution of $VT'M$. Let $VT'\widetilde{M}^{\perp}$ be an orthogonal distribution to $VT'\widetilde{M}$ in $VT'M$ with respect to Hermitian metric structure $G^v = g_{i\bar{j}} d\eta^i \otimes d\bar{\eta}^j$ and by $\{N_a = B^i_a \frac{\partial}{\partial \eta^i}\}_{a=\overline{2,n}}$ we denote a basis in $VT'\widetilde{M}^{\perp}$, which we can choose to be orthonormal. Thus we have, $g_{i\bar{j}} B^i B^{\bar{j}}_{\bar{a}} = 0$ and $g_{i\bar{j}} B^i_a B^{\bar{j}}_{\bar{b}} = \delta_{ab}$. The frame $\mathcal{R} = [B^i \ B^i_a]$ is a moving frame along $\widetilde{M}$ and let $\mathcal{R}^{-1} = [B_i \ B^a_i]$ be the inverse matrix of $\mathcal{R}$. It follows that

$$B_i B^i = 1 \ ; \quad B_i B^i_a = 0 \ ; \quad B^a_i B^i = 0 \ ; \quad B^a_i B^i_b = \delta^a_b. \tag{2.8}$$

It can be easily checked that $\frac{\partial}{\partial \eta^i} = B_i \frac{d}{d\theta} + B^a_i N_a$ and also that we have the following links between the dual bases

$$dz^i = B^i dw \ ; \quad d\eta^i = \dot{B}^i_0 dw + B^i d\theta. \tag{2.9}$$

Let N^i_j be the coefficients of the Chern-Finsler complex nonlinear connection, like in the above section, and $\delta\eta^i = d\eta^i + N^i_j dz^j$ be its adapted dual cobasis. An *induced* (*c.n.c*) on $\widetilde{M}$ is defined by $\delta\theta = B_i \delta\eta^i$, where $\delta\theta = d\theta + N dw$. From the general theory, [Mu], p. 130, it results

Proposition 2.1 $N = B_i(\ \dot{B}^i_0 + B^j N^i_j)$.

For the adapted frames $\frac{\delta}{\delta w} = \frac{d}{dw} - N \frac{d}{d\theta}$ and $\frac{\delta}{\delta z^i} = \frac{\partial}{\partial z^i} - N^j_i \frac{\partial}{\partial \eta^j}$, according to Proposition 5.4.2 from [Mu], we have

Proposition 2.2 *i)* $dz^i = B^i dw \ ; \quad \delta\eta^i = B^i \delta\theta + B^i_a M^a dw \ ;$
ii) $\frac{\delta}{\delta w} = B^i \frac{\delta}{\delta z^i} + B^i_a M^a \frac{\partial}{\partial \eta^i} \ ; \quad \frac{d}{d\theta} = B^i \frac{\partial}{\partial \eta^i},$ *where* $M^a = B^a_j(\dot{B}^j_0 + B^k N^j_k)$.

It is proved in [Mu] that the induced (*c.n.c.*) coincides with the intrinsic (*c.n.c.*) of $(\widetilde{M}, \tilde{F})$, that is $N = g^{-1} d^2 \tilde{L} / dw d\bar{\theta}$, where $\tilde{L} = \tilde{F}^2$ and $g = g_{i\bar{j}} B^i \bar{B}^j$ is the induced metric on $(\widetilde{M}, \tilde{F})$.

Let $D\Gamma(N) = (L^i_{jk}, 0, C^i_{jk}, 0)$ be the Chern-Finsler linear connection as in (2.4).

According to Theorem 5.4.4, the induced tangent connection of Chern-Finsler linear connection is $\tilde{D}\Gamma(\tilde{N}) = (L, 0, C, 0)$, where

$$L = g^{-1} \frac{\delta g}{\delta w} \ ; \quad C = g^{-1} \frac{dg}{d\theta}. \tag{2.10}$$

Like in classical theory the Gauss and Weingarten operators can be introduced here (see [Mu2]).

Further on we make a remark about the induced holomorphic sectional curvature. Recall that, [A-P,Mu], the holomorphic sectional curvature of a complex Finsler space $(M, L = F^2)$ is given by

$$K_F(z, \eta) = \frac{2}{L^2} G(\Omega(\chi, \bar{\chi})\chi, \bar{\chi}) \tag{2.11}$$

where G is the Hermitian metric structure, Ω is the curvature form of Chern-Finsler connection and $\chi = \eta^k \frac{\delta}{\delta z^k}$ is the horizontal lift of the radial vertical vector $\eta = \eta^k \frac{\partial}{\partial \eta^k}$.

K_F in η direction is written as a function of Ricci tensor as follows,

$$K_F(z,\eta) = \frac{2}{L^2} R_{\bar{j}k} \bar{\eta}^j \eta^k \ , \quad \text{where} \ \ R_{\bar{j}k} = -g_{l\bar{j}} \frac{\delta N_k^l}{\delta \bar{z}^h} \bar{\eta}^h. \tag{2.12}$$

In [Mu1] we found for a general holomorphic subspace $(\widetilde{M}, \tilde{F})$ the relationship between K_F and intrinsic holomorphic sectional curvature $\tilde{K}_{\tilde{F}}$. Particularizing this result it follows that $\tilde{K}_{\tilde{F}}$ in $\tilde{u} = (w, \theta)$, with θ a tangent direction of the holomorphic curve, and K_F in $u = (z(w), \eta(w,\theta))$ are connected by

$$\tilde{K}_{\tilde{F}}(\tilde{u}) = K_F(u) - \frac{2}{L^2} B_{\bar{a}}^{\bar{\imath}} B_{\bar{h}}^{\bar{a}} \{Q_{k\bar{\imath}}^m + \rho_{k\bar{\imath}}^m\} g_{m\bar{j}} \bar{\eta}^j \bar{\eta}^h \eta^k \ , \tag{2.13}$$

where $Q_{k\bar{\imath}}^m = \frac{\delta}{\delta \bar{z}^i}(N_k^m)$ and $\rho_{k\bar{\imath}}^m = \frac{\partial}{\partial \bar{\eta}^i}(N_k^m)$.

For instance, if $(\widetilde{M}, \tilde{F})$ is a Riemann surface of a locally Minkowsky space (M, F), then there exists a local chart in any point u in which $N_k^m = 0$ and consequently, in such charts, the intrinsic holomorphic curvature of $(\widetilde{M}, \tilde{F})$ coincides with that of (M, F).

Fore more details and other problems, like totally geodesic holomorphic curves, see [Mu2].

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