PAIRS OF METRICAL FINSLER STRUCTURES AND FINSLER CONNECTIONS COMPATIBLE TO THEM

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We consider a pair of metrical Finsler structure $g_{ij}(x,y), s_{ij}(x,y), (x,y) \in TM,\ i,j = \overline{1,n}$, $\dim M = n$ and we investigate the cases in which is possible to find Finsler connections compatible to them: $rank\,\|g_{ij}(x,y)\| = n$, $rank\,\|s_{ij}(x,y)\| = n-k,\ k \in \{0,1,...,n-1\}, \forall (x,y) \in TM \setminus \{0\}$.

1 Metrical Finsler structures and metrical Finsler connections ([7])

Let M be an $n-$dimensional differentiable manifold and $x = (x^i)$ and $y = (y^i)$ denote a point of M and a supporting element respectively. We put $\partial_i = \partial/\partial x^i,\ \dot{\partial}_i = \partial/\partial y^i, (i = 1,2,...,n)$.

Let $g_{ij}(x,y) = \left(\dot{\partial}_i \dot{\partial}_j F^2\right)/2$ be a Finsler metric and $N\left(N^i{}_j\right)$ a nonlinear connection, which us given the adapted basis $\left\{\delta_i,\ \dot{\partial}_i\right\}$ of the tangent bundle $TM =: HM \oplus VM$:

$$\delta_i = \frac{\delta}{\delta x^i} = \partial_i - N^j{}_i \dot{\partial}_j. \tag{1.1}$$

We denote $\{dx^i, dy^i\}$ the dual basis of adapted basis, where

$$\delta y^i = dy^i + N^i{}_j dx^j. \tag{1.2}$$

We shall express a Finsler connection $F\Gamma$ in terms of its coefficients as $F\Gamma = (N^j{}_k, F^i{}_{jk}, C^i{}_{jk})$, (cf. with M. Matsumoto [4], R. Miron [7] and E. Stoica [13]). A Finsler connection having a fixed nonlinear connection N is also denoted by $F\Gamma(N) = (F^i{}_{jk}, C^i{}_{jk})$. And the respective h- and v-covariant derivatives are denoted by short and long bars, *e.g.*, $g_{ij\shortmid k}, g_{ij|k}$ (with respect to $F\Gamma$), $g_{ij\overset{\circ}{\shortmid}k}, g_{ij\overset{\circ}{|}k}$ (with respect to $F\overset{\circ}{\Gamma}$), etc.

Given a Finsler metric g_{ij}, a Finsler connection $F\Gamma$ is called **metrical**, if it satisfies

$$g_{ij\shortmid k} = 0, \quad g_{ij|k} = 0. \tag{1.3}$$

For a Finsler metric g_{ij}, we have so-called Obata's operators, [10]:

$$\underset{1}{\Lambda}{}^{pq}_{ij} = \frac{1}{2}\left(\delta^p_i \delta^q_j - g_{ij} g^{pq}\right), \quad \underset{2}{\Lambda}{}^{qp}_{ij} = \frac{1}{2}\left(\delta^p_i \delta^q_j + g_{ij} g^{pq}\right), \tag{1.4}$$

where $(g^{ij}) = (g_{ij})^{-1}$. Then we have

Theorem 1.1 *Let $F\overset{\circ}{\Gamma}(N) = \left(\overset{\circ}{\Gamma}{}^i{}_{jk}, \overset{\circ}{C}{}^i{}_{jk}\right)$ be a fixed Finsler connection. For a Finsler metric g_{ij}, we define tensor fields $U^i{}_{jk}, \tilde{U}^i{}_{jk}$ by*

$$U^i{}_{jk} = -\frac{1}{2} g^{ir} g_{rj\overset{\circ}{\shortmid}k}, \quad \tilde{U}^i{}_{jk} = -\frac{1}{2} g^{ir} g_{rj\overset{\circ}{|}k}. \tag{1.5}$$

Then a Finsler connection $F\Gamma(N) = (F^i{}_{jk}, C^i{}_{jk})$ is metrical, if and only if the difference tensor fields $B^i{}_{jk}, D^i{}_{jk}$ given by

$$F^i{}_{jk} = \overset{\circ}{F}{}^i{}_{jk} - B^i{}_{jk}, \quad C^i{}_{jk} = \overset{\circ}{C}{}^i{}_{jk} - D^i{}_{jk}, \tag{1.6}$$

are solutions of the equations

$$\underset{2}{\Lambda}{}^{ip}_{qj} B^q{}_{pk} = U^i{}_{jk}, \quad \underset{2}{\Lambda}{}^{ip}_{qj} D^q{}_{pk} = \tilde{U}^i{}_{jk}. \tag{1.7}$$

Conform with Obata's theory, [10], the above equations have solutions and their general forms are given by

Theorem 1.2 (R. Miron, [7]) Let $F\overset{\circ}{\Gamma}(N) = \left(\overset{\circ}{F}{}^i{}_{jk}, \overset{\circ}{C}{}^i{}_{jk}\right)$ be a fixed Finsler connection. For a Finsler metric g_{ij}, there exists a metrical Finsler connection $F\Gamma(N) = (F^i{}_{jk}, C^i{}_{jk})$ and the set of all such connections is given by

$$\begin{aligned} F^i{}_{jk} &= \overset{\circ}{F}{}^i{}_{jk} + \frac{1}{2} g^{ir} g_{rj \overset{\circ}{|} k} + \underset{1}{\Lambda}{}^{ip}_{qj} X^q{}_{pk}, \\ C^i{}_{jk} &= \overset{\circ}{C}{}^i{}_{jk} + \frac{1}{2} g^{ir} g_{rj \overset{\circ}{\|} k} + \underset{1}{\Lambda}{}^{ip}_{qj} Y^q{}_{pk}, \end{aligned} \tag{1.8}$$

where $X^i{}_{jk}, Y^i{}_{jk}$ are arbitrary Finsler tensor fields.

2 Finsler connections compatible with a pair of Finsler metrics

Let g_{ij} and s_{ij} be two given Finsler metrics. A Finsler connection is called **compatible** with the pair (g_{ij}, s_{ij}), if it is metrical with respect to both g_{ij} and s_{ij}:

$$g_{ij|k} = 0, \quad g_{ij\|k} = 0, \quad s_{ij|k} = 0, \quad s_{ij\|k} = 0. \tag{2.1}$$

We define Obata's operators by (1.4) and

$$\underset{1}{O}{}^{pq}_{ij} = \frac{1}{2}\left(\delta^p_i \delta^q_j - s_{ij} s^{pq}\right), \quad \underset{2}{O}{}^{pq}_{ij} = \frac{1}{2}\left(\delta^p_i \delta^q_j + s_{ij} s^{pq}\right), \tag{2.2}$$

where $(s^{ij}) = (s_{ij})^{-1}$. Then we have

Theorem 2.1 *Let $F\overset{\circ}{\Gamma}(N) = \left(\overset{\circ}{F}{}^i{}_{jk}, \overset{\circ}{C}{}^i{}_{jk}\right)$ be a fixed Finsler connection. For a pair of Finsler metrics (g_{ij}, s_{ij}) we define Finsler tensor fields $U^i{}_{jk}, \tilde{U}^i{}_{jk}, V^i{}_{jk}, \tilde{V}^i{}_{jk}$ by (1.5) and*

$$V^i{}_{jk} = -\frac{1}{2} s^{ir} s_{rj \overset{\circ}{|} k}, \quad \tilde{V}^i{}_{jk} = -\frac{1}{2} s^{ir} s_{rj \overset{\circ}{\|} k}. \tag{2.3}$$

Then a Finsler connection $F\Gamma(N) = (F^i{}_{jk}, C^i{}_{jk})$ is compatible with the pair (g_{ij}, s_{ij}), if and only if the difference tensor fields $B^i{}_{jk}, D^i{}_{jk}$ given by (2.2) *are solutions of the equations* (2.3) *and following equations*

$$\underset{2}{O}{}^{ip}_{qj} B^q{}_{pk} = V^i{}_{jk}, \ \underset{2}{O}{}^{ip}_{qj} D^q{}_{pk} = \tilde{V}^i{}_{jk}. \tag{2.4}$$

It is complicated to solve the above equations.

We shall show the case when the equations have solutions.

A pair of two Finsler metrics g_{ij}, s_{ij} is called **natural**, if there exists a nonvanishing Finsler function $\mu(x, y)$ such that

$$g_{ip}g_{jq}s^{pq} = \mu s_{ij}, \tag{2.5}$$

or equivalently, if the commutativities

$$\underset{\alpha}{\Lambda}{}^{ip}_{qj}\underset{\beta}{O}{}^{qr}_{tp} = \underset{\beta}{O}{}^{ip}_{qj}\underset{\alpha}{\Lambda}{}^{qr}_{tp}, \quad (\alpha, \beta = 1, 2), \tag{2.6}$$

hold. Then, we have

Proposition 2.1 *All the commutativities* (2.6) *hold if any one of them holds.*

Proposition 2.2 *Let* (g_{ij}, s_{ij}) *be a natural pair of Finsler metrics. If there exists a Finsler connection compatible with the pair, the function* μ *in* (2.5) *is constant.*

Proof. The equations (2.1) are equivalent by the following equations:

$$g^{ij}{}_{\shortmid k} = 0, \quad g^{ij}\,|_k = 0, \quad s^{ij}{}_{\shortmid k} = 0, \quad s^{ij}\,|_k = 0. \tag{2.7}$$

By (2.1) and (2.1′) we have $\mu_{\shortmid k}s_{ij} = 0$, $\mu\,|_k\, s_{ij} = 0$, which are reduced to $\mu_{\shortmid k} = 0$, $\mu\,|_k = 0$ because $s_{ij}s^{ij} = n \neq 0$. Hence the nonvanishing function μ is constant.

Proposition 2.3 *Let* g_{ij} *be a Finsler metric. There exists a Finsler metric* s_{ij} *such that the pair* (g_{ij}, s_{ij}) *is natural by a constant* $\mu = \varepsilon c^2\,(\varepsilon = \pm 1, c > 0)$, *if and only if there exists a Finsler tensor field* $t^i{}_j$ *of type* $(1, 1)$ *satisfying*

$$\varepsilon t^i{}_r t^r{}_j = \delta^i_j, \quad \varepsilon g_{pq}t^p{}_i t^q{}_j = g_{ij}. \tag{2.8}$$

The correspondence between $t^i{}_j$ and s_{ij} in Proposition 2.3 is given by

$$t^i{}_j = cg^{ir}s_{rj}, \quad s_{ij} = \frac{1}{c}g_{ir}t^r{}_j. \tag{2.9}$$

Remark 2.1 If $\varepsilon = -1$, then $\mu = -c^2$ and $t^i{}_j$ is an almost complex Finsler structure $f^i{}_j : f^2 = -I$, $(n = 2m)$. In this case, the natural pair (g_{ij}, s_{ij}) is called of **elliptical type,** or a **(g, f, −1)−structure** (cf. with Gh. Atanasiu [12], Gh. Atanasiu, M. Hashiguchi, R. Miron [3]), or an **anti-Hermitian structure**:

$$f^i{}_r f^r{}_j = -\delta^i_j, \quad g_{pq}f^p{}_i f^q{}_j = -g_{ij}. \tag{2.10}$$

Remark 2.2 If $\varepsilon = +1$, then $\mu = c^2$ and $t^i{}_j$ is an almost product Finsler structure $p^i{}_j : p^2 = I$. In this case, the natural pair (g_{ij}, s_{ij}) is called of **hyperbolical type,** or a **(g, p, +1)−structure** (see [12], [3]:

$$p^i{}_r p^r{}_j = \delta^i_j, \quad g_{rt}p^r{}_i p^t{}_j = g_{ij}. \tag{2.11}$$

Using Proposition 2.3 we can show that for a natural pair (elliptic or hyperbolic) with a constant $\mu \neq 0$ the equations (2.3) and (2.4) have solutions and their general forms are given by

Theorem 2.2 *Let* $F\overset{\circ}{\Gamma}(N) = \left(\overset{\circ}{F}{}^i{}_{jk}, \overset{\circ}{C}{}^i{}_{jk}\right)$ *be a fixed Finsler connection. For a natural pair with a constant* $\mu \neq 0$ *of Finsler metric* g_{ij}, s_{ij}, *there exists a Finsler connection* $F\Gamma(N) = (F^i{}_{jk}, C^i{}_{jk})$ *compatible with the pair and the set of all such connections is given by*

$$\begin{aligned} F^i{}_{jk} &= \overset{\circ}{F}{}^i{}_{jk} + \frac{1}{2}\left(g^{ir}g_{rj\overset{\circ}{\shortmid}k} + \underset{1}{\Lambda}{}^{ip}_{qj}s^{qt}s_{tp\overset{\circ}{\shortmid}k} \right) + \underset{1}{\Lambda}{}^{ip}_{qj}\underset{1}{O}{}^{qr}_{tp}X^t{}_{rk}, \\ C^i{}_{jk} &= \overset{\circ}{C}{}^i{}_{jk} + \frac{1}{2}\left(g^{ir}g_{rj}\overset{\circ}{|}_k + \underset{1}{\Lambda}{}^{ip}_{qj}s^{qt}s_{tp}\overset{\circ}{|}_k \right) + \underset{1}{\Lambda}{}^{ip}_{qj}\underset{1}{O}{}^{qr}_{tp}Y^t{}_{rk}, \end{aligned} \tag{2.12}$$

where $X^i{}_{jk}, Y^i{}_{jk}$ *are arbitrary Finsler tensor fields.*

3 The case of Finsler metric with an additional structure

The previous results for a pair of Finsler metrics $g_{ij}(x,y), s_{ij}(x,y)$ are generalized to the case $s_{ij}(x,y)$ is degenerate.

Let a Finsler space (M, g_{ij}) admit a symmetric and degenerate Finsler tensor field $s_{ij}(x,y)$:

$$s_{ij} = s_{ji} \tag{3.1}$$

$$rank\,(s_{ij}) = n - k, \tag{3.2}$$

where k is an integer and $0 < k < n$. Then (M, g_{ij}) is called to have an **additional structure of index k**. The case of a Finsler metric $s_{ij}\,(x,y)$ is contained in the following duscussions as the exceptional case $k = 0$.

The matrix (g_{ij}) has the inverse $\left(g^{jk}\right)$, but the matrix (s_{ij}) is not regular. So we shall construct some matrix $\left(s^{jk}\right)$ which plays the role similar to the inverse matrix. (see, V. Oproiu [11], [12]). Because (g_{ij}) is positive-definite, then on each local chart there are exaktly k independent Finsler vector fields $\xi^i_a\,(a = 1, ..., k)$ with the properties

$$s_{ij}\xi^i_a = 0, \quad g_{ij}\xi^i_a\xi^j_b = \delta_{ab} \quad (a, b = 1, ..., k)\,. \tag{3.3}$$

Then we define local Finsler covector fields $\eta^a_i\,(a = 1, ..., k)$ by

$$\eta^a_i = g_{ij}\xi^j_a. \tag{3.4}$$

If we define local Finsler tensor fields $l^i{}_j$ and $m^i{}_j$ by

$$l^i{}_j = \sum_a \xi^i_a\eta^a_i, \quad m^i{}_j = \delta^i_j - l^i{}_j, \tag{3.5}$$

then $l^i{}_j$ and $m^i{}_j$ are independent on the choice of ξ^i_a and globally defined as the respective projectors on the kernel $\mathbf{K}$ of the mapping $s_{ij} :\ \xi^j_a \longrightarrow s_{ij}\xi^j_a$ and the orthogonal $\mathbf{H}$ to $\mathbf{K}$ with respect to g_{ij}. Then a global Finsler tensor field s^{jk} is uniquely determined from (g_{ij}, s_{ij}) by

$$s_{ij}s^{jk} = m^k{}_i, \quad l^i{}_j s^{jk} = 0. \tag{3.6}$$

A Finsler connection of a Finsler space (M, g_{ij}) with an additional structure s_{ij} is called **compatible** with the pair (g_{ij}, s_{ij}) , if it satisfies (2.1) . Then the condition that a Finsler connection $F\Gamma$ is compatible with the pair (g_{ij}, s_{ij}) is given by Theorem 2.1, if we define $V^i{}_{jk}, \tilde{V}^i{}_{jk}$ by

$$\begin{aligned} V^i{}_{jk} &= -\frac{1}{2}\left(s^{ir}s_{rj\overset{\circ}{|}k} + 3l^i{}_t l^t{}_{j\overset{\circ}{|}k} - l^i{}_{j\overset{\circ}{|}k}\right), \\ \tilde{V}^i{}_{jk} &= -\frac{1}{2}\left(s^{ir}s_{rj}\overset{\circ}{|}_k + 3l^i{}_t l^t{}_j\overset{\circ}{|}_k - l^i{}_j\overset{\circ}{|}_k\right), \end{aligned} \tag{3.7}$$

and Obata's operators $\underset{\alpha}{O}{}^{pq}_{ij}\ (\alpha = 1, 2)$ by

$$\begin{aligned} \underset{1}{O}{}^{pq}_{ij} &= \frac{1}{2}\left(\delta^p_i\delta^q_j - \delta^p_i l^q_j - l^p_i\delta^q_j + 3l^p_i l^q_j - s_{ij}s^{pq}\right), \\ \underset{2}{O}{}^{pq}_{ij} &= \frac{1}{2}\left(\delta^p_i\delta^q_j + \delta^p_i l^q_j + l^p_i\delta^q_j - 3l^p_i l^q_j + s_{ij}s^{pq}\right), \end{aligned} \tag{3.8}$$

and impose on the $B^i{}_{jk}$ and $D^i{}_{jk}$ the additional conditions:

$$
\begin{aligned}
l^r{}_i s_{tj} B^t{}_{rk} &= -l^r{}_i s_{rj\overset{\circ}{|}k}, \quad l^r{}_i s_{tj} B^t{}_{rk} = -l^r{}_i s_{rj}\overset{\circ}{|}_k, \\
l^i{}_t m^r{}_j B^t{}_{rk} &= -l^i{}_t l^t{}_{j\overset{\circ}{|}k}, \quad l^i{}_t m^r{}_j D^t{}_{rk} = -l^i t l^t{}_j\overset{\circ}{|}_k.
\end{aligned}
\tag{3.9}
$$

If we define the naturality of a pair (g_{ij}, s_{ij}) by (2.5), or equivalently (2.6) where $\underset{\alpha}{O}{}^{pq}_{ij}$ are defined by (3.8), then Propositions 2.1 and 2.2 still hold. Corresponding to Proposition 2.3, the condition that a Finsler space (M, g_{ij}) admits an additional structure s_{ij} of index k such that the pair (g_{ij}, s_{ij}) is natural by a constant $\mu = \varepsilon c^2$ $(\varepsilon = \pm 1, c > 0)$ is given by the existence of a Finsler tensor field $t^i{}_j$ of type $(1,1)$, k Finsler vector fields ξ^i_a $(a = 1, ..., k)$ and k Finsler covector fields η^a_i $(i = 1, ..., k)$ satisfying

$$
\begin{aligned}
\varepsilon t^i{}_r t^r{}_j = \delta^i_j - \xi^i_a \eta^a_j, \quad \varepsilon g_{pq} t^p{}_i t^q{}_j = g_{ij} - \sum_a \eta^a_i \eta^a_j, \\
\eta^a_i t^i{}_j = 0, \quad t^i{}_j \xi^j_a = 0, \ \eta^a_i \quad \xi^i_b = \delta^a_b.
\end{aligned}
\tag{3.10}
$$

Remark 3.1 If $\varepsilon = -1$, then $\mu = -c^2$ and $t^i{}_j$ is an degenerate almost complex Finsler structure $f^i{}_j(x, y)$:

$$
\begin{aligned}
f^i{}_r f^r{}_j = -\delta^i_j + \xi^i_a \eta^a_j, \quad g_{pq} f^p{}_i f^q{}_j = -g_{ij} + \sum_a \eta^a_i \eta^a_j, \\
\eta^a_i f^i{}_j = 0, \quad f^i{}_j \xi^j_a = 0, \quad \eta^a_i \xi^i_b = \delta^a_b.
\end{aligned}
\tag{3.11}
$$

In this case we have a $(g, f, \xi, \eta, -1)$ −structure, [10], [9].

Remark 3.2 If $\varepsilon = +1$, then $\mu = c^2$ and $t^i{}_j$ is an degenerate almost product Finsler structure $p^i{}_j(x, y)$:

$$
\begin{aligned}
p^i{}_r p^r{}_j = \delta^i_j - \xi^i_a \eta^a_j, \quad g_{rt} p^r{}_i p^t{}_j = g_{ij} - \sum_a \eta^a_i \eta^a_j, \\
\eta^a_i p^i{}_j = 0, \quad p^i{}_j \xi^j_a = 0, \quad \eta^a_i \xi^i_b = \delta^a_b.
\end{aligned}
\tag{3.12}
$$

and we have a $(g, p, \xi, \eta, +1)$ −structure, [3], [9].

The existence and arbitrariness of Finsler connections compatible with a pair $(g_{ij},\ s_{ij})$ with a constant $\mu \neq 0$, is given by

Theorem 3.1 Let $F\Gamma\left(\overset{\circ}{N}\right) = \left(\overset{\circ}{F}{}^i{}_{jk}, \overset{\circ}{C}{}^i{}_{jk}\right)$ be a fixed Finsler connection. There exists a Finsler connection $F\Gamma(N) = (F^i{}_{jk}, C^i{}_{jk})$ compatible with the pair and the set of all such connections is given by

$$
\begin{aligned}
F^i{}_{jk} &= \overset{\circ}{F}{}^i{}_{jk} + \frac{1}{2}\left[g^{ir} g_{rj\overset{\circ}{|}k} + \underset{1}{\Lambda}{}^{ip}_{qj}\left(s^{qt} s_{tp\overset{\circ}{|}k} + 3 l^q{}_t l^t{}_{p\overset{\circ}{|}k} - l^q{}_{p\overset{\circ}{|}k}\right)\right] + \underset{1}{\Lambda}{}^{ip}_{qj} \underset{1}{O}{}^{qr}_{tp} X^t{}_{rk}\,, \\
C^i{}_{jk} &= \overset{\circ}{C}{}^i{}_{jk} + \frac{1}{2}\left[g^{ir} g_{rj}\overset{\circ}{|}_k + \underset{1}{\Lambda}{}^{ip}_{qj}\left(s^{qt} s_{tp}\overset{\circ}{|}_k + 3 l^q{}_t l^t{}_p\overset{\circ}{|}_k - l^q{}_p\overset{\circ}{|}_k\right)\right] + \underset{1}{\Lambda}{}^{ip}_{qj} \underset{1}{O}{}^{qr}_{tp} Y^t{}_{rk}.
\end{aligned}
\tag{3.13}
$$

where $\underset{1}{O}{}^{pq}_{ij}$ is given by (3.8) and $X^i{}_{jk}, Y^i{}_{jk}$ are arbitrary Finsler tensor fields.

Lastly, it is noted whether the naturality is necessary in order that the system of equations (2.3), (2.4), (3.9) with unknowns $B^i{}_{jk}, D^i{}_{jk}$ has a solution is an open problem.

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