GENERALIZED–ANALYTICAL FUNCTIONS OF POLY–NUMBER VARIABLE

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We introduce the notion of the generalized-analytical function of the poly–number variable, which is a non–trivial generalization of the notion of analytical function of the complex variable and, therefore, may turn out to be fundamental in theoretical physical constructions. As an example we consider in detail the associative-commutative hypercomplex numbers H_4 and an interesting class of corresponding functions.

1. Introduction

Let M_n be an n-dimensional elementary manifold and P_n denote the system of n-dimensional associative-commutative hypercomplex numbers (poly-numbers, n-numbers), and a one-to-one correspondence between the sets be assigned. Under these conditions, we choose in P_n the basis

$$e_1, e_2, ..., e_n; \quad e_i e_j = p_{ij}^k e_k, \tag{1}$$

$$X = x^1 \cdot e_1 + x^2 \cdot e_2 + \cdots + x^n e_n \in P_n, \tag{2}$$

where $e_1, e_2, ..., e_n$ – symbolic elements, p_{ij}^k stand for characteristic real numbers, and $x^1, x^2, ..., x^n$ – real coordinates with respect to the basis $(e_1 \equiv 1, e_2, \ldots, e_n)$. Obviously, the numbers $x^1, x^2, ..., x^n$ can be used not only as the coordinates in P_n, but also as coordinates in the manifold M_n , so that $(x^1, x^2, ..., x^n) \in M_n$. Though in M_n we can go over to any other curvilinear reference frame, the reference frame $\{x^i\}$, as being built by the help the basis of poly–numbers and a fixed one-to-one correspondence $M_n \leftrightarrow P_n$, ought to be considered preferable (as well as any other reference frame connected with this by non-degenerate linear transformation). The poly–number algebraic operations induce the same operations in the elementary manifold (formally) and in the tangent space at any point of the manifold (informally). Accordingly, the tangent spaces to M_n are isomorphic to P_n.

The function

$$F(X) := f^1(x^1, \cdots, x^n)e_1 + \cdots + f^n(x^1, \cdots, x^n)e_n \tag{3}$$

of the poly–number variable, where f^i are sufficiently smooth functions of n real variables, will be considered to be a vector (contravariant) field in M_n. Hence, apart from addition and multiplication by number, any operation of multiplication of vector fields

$$f_{(3)}^k = f_{(1)}^i \cdot f_{(2)}^j \cdot p_{ij}^k \tag{4}$$

can also defined in M_n. It is useful but not obligatory to consider the space M_n to be the main ("the examined") object and the space P_n to be a sort of an instrument with the help of which the space M_n is "examined". In the general case the parallel transportation of a vector in the space P_n does not correspond to the "parallel transportation" of the same vector in the space M_n, so that for a due definition of absolute differential (or the covariant derivative) we are to have the connection objects or the quantities which may replace them. If we avoid introducing the pair $\{M_n, P_n\}$, restricting the treatment only to associative-commutative hypercomplex numbers, then it is natural to introduce the definitions

$$dX := dx^i \cdot e_i \tag{5}$$

and

$$dF(X) := F(X + dX) - F(X) = \frac{\partial f^i}{\partial x^k} \cdot e_i \cdot dx^k. \tag{6}$$

The function $F(X)$ of poly-number variable X is called *analytical*, if such a function $F'(X)$ exists that

$$dF(X) = F'(X) \cdot dX, \tag{7}$$

where the multiplication in the right-hand part means the poly-number operation. From (7) it follows that

$$\frac{\partial f^i}{\partial x^k} = p^i_{kj} \cdot f'^j. \tag{8}$$

Since with respect to the basis e_i with the components $e_1 = 1$ the equalities

$$p^i_{1j} = \delta^i_j \tag{9}$$

hold, we have

$$f'^i = \frac{\partial f^i}{\partial x^1}. \tag{10}$$

Inserting (10) in (8) yields the Cauchy-Riemann relations

$$\frac{\partial f^i}{\partial x^1} - p^i_{kj} \cdot \frac{\partial f^j}{\partial x^1} = 0 \tag{11}$$

for the functions under study. The number $n(n-1)$ of these relations is growing quicker that the number n of components of analytical function. This leads to the *functional restriction* of the set of such functions at $n > 2$. The present work is just attempting to elaborate a non-trivial extension of the notion of analytical function of poly-number variable subject to the condition that number of the Cauchy–Riemann–type conditions does nor exceed the number of unknown function–components. The first step in this direction has been made above when introducing the pair $\{M_n, P_n\}$. Therefore it seems natural to replace the differential 6) by means of the absolute differential

$$DF(X) := \bigtriangledown_k f^i \cdot e_i \cdot dx^k, \tag{12}$$

where

$$\bigtriangledown_k f^i := \frac{\partial f^j}{\partial x^k} + \Gamma^i_{kj} \cdot f^j \tag{13}$$

is the covariant derivative, and Γ^i_{kj} means "the connection coefficients". Instead of the formulas (8) and (10) we get

$$\bigtriangledown_k f^i = p^i_{kj} \cdot f'^j \tag{14}$$

and

$$f'^i = \bigtriangledown_1 \cdot f^i, \tag{15}$$

and the Cauchy–Riemann conditions take on the form

$$\bigtriangledown_k f^i - p^i_{kj} \cdot \bigtriangledown_1 f^j = 0. \tag{16}$$

Of course, "the connection objects" Γ^i_{kj} in the formula (13) are not obligatory to be uniform for all the set of functions obeying the conditions (16).

2. Definitions and basic implications

Let us call the function $F(X)$ *generalized–analytical*, if such a function $F'(X)$ exists that

$$\widetilde{D}F(X) = F'(X) \cdot dX, \tag{17}$$

where

$$\widetilde{D}F(X) \equiv \widetilde{\nabla}_k f^i \cdot e_i \cdot dx^k \tag{18}$$

and the definition

$$\widetilde{\nabla}_k f^i := \frac{\partial f^i}{\partial x^k} + \gamma^i_k \tag{19}$$

has been used. It is assumed that under the transition from one (curvilinear) coordinate system to another coordinate system the involved objects γ^i_k are transformed according to the law

$$\gamma^{i'}_{k'} = \frac{\partial x^k}{\partial x^{k'}} \cdot \frac{\partial x^{i'}}{\partial x^i} \cdot \gamma^i_k - \frac{\partial x^k}{\partial x^{k'}} \cdot \frac{\partial^2 x^{i'}}{\partial x^k \partial x^i} \cdot f^i. \tag{20}$$

It will be noted that such a definition entails that $\widetilde{\nabla}_k f^i$ behaves like a tensor. The quantities γ^i_k will be called the *gamma-objects*. In general we do not assume the relations

$$\gamma^i_k = \Gamma^i_{kj} \cdot f^j \tag{21}$$

with a single "connection object" Γ^i_{kj} for generalized-analytical functions. It would be more precise to say of the pair $\{f^i, \gamma^i_k\}$, such that the analytical function of poly-number variable is the pair $\{f^i, 0\}$, but this pair transform to the pair $\{f^i, \gamma^{i'}_{k'} \neq 0\}$ under going over from the special coordinate system to another curvilinear one.

From the definition of generalized-analytical functions it follows that

$$\widetilde{\nabla}_k f^i = p^i_{kj} \cdot f'^j \tag{22}$$

and

$$f'^j = \widetilde{\nabla}_1 f^i; \tag{23}$$

the respective generalized Cauchy–Riemann relations take on the form

$$\widetilde{\nabla}_k f^j - p^i_{kj} \widetilde{\nabla}_1 f^j = 0. \tag{24}$$

The number of unknown functions in the pair $\{f^i, \gamma^i_k\}$ equals $n + n^2 = n(n+1)$, – which is more than the number $n(n-1)$ of the generalized Cauchy-Riemann relations (24). Thus, to use the notion of generalized-analytical function in theoretical-physical constructions it is necessary to additionally establish and formulate the set of requirements (possibly one requirement) which, when used in conjunction with the notion of generalized-analytical function, would lead unambiguously to equations of some field of physical meaning. Usually, they are n partial differential equations of second order for n independent function-component field.

If $\{f^i_{(1)}, \gamma^i_{(1)k}\}$ and $\{f^i_{(2)}, \gamma^i_{(2)k}\}$ – two generalized-analytical functions, then their arbitrary linear sum with real coefficients α, β is a generalized-analytical function. This ensues directly from the definition, on using also the formulae (22)–(24) and (20). Thus, we have

$$\alpha \cdot \{f^i_{(1)}, \gamma^i_{(1)k}\} + \beta \cdot \{f^i_{(2)}, \gamma^i_{(2)k}\} = \{\alpha \cdot f^i_{(1)} + \beta \cdot f^i_{(2)}, \alpha \cdot \gamma^i_{(1)k} + \beta \cdot \gamma^i_{(2)k}\}. \tag{25}$$

Now, let us consider the poly-number product of two generalized-analytical functions $f^i_{(1)}$ and $f^j_{(2)}$:

$$f^k_{(3)} = f^i_{(1)} \cdot f^j_{(2)} \cdot p^k_{ij} \tag{26}$$

and try to find the object $\gamma^i_{(3)k}$ such that the pair $\{f^i_{(3)}, \gamma^i_{(3)k}\}$ be generalized-analytical function. To this end we formally differentiate the left and right parts of (26) with respect to x^k and use the formula (22), obtaining

$$\frac{\partial f^i_{(3)}}{\partial x^k} + \gamma^i_{(3)k} = p^{i_1}_{kj} p^i_{i_1 i_2} f'^j_{(1)} f'^{j_2}_{(2)} + p^{i_2}_{kj} p^i_{i_1 i_2} f^{i_1}_{(1)} f'^j_{(2)}. \tag{27}$$

Owing to the formula

$$p^r_{im} \cdot p^m_{kj} = p^r_{km} \cdot p^m_{ij} \tag{28}$$

(which is an implication of the properties of associativity and commutativity of poly-numbers), we can write

$$\frac{\partial f^i_{(3)}}{\partial x^k} + \gamma^i_{(3)k} = p^i_{kj} p^j_{i_1 i_2} (f'^{i_1}_{(1)} f^{i_2}_{(2)} + f^{i_1}_{(1)} f'^{i_2}_{(2)}), \tag{29}$$

where

$$\gamma^i_{(3)k} = p^i_{i_1 i_2} \cdot (\gamma^{i_1}_{(1)k} f^{i_2}_{(2)} + f^{i_1}_{(1)} \gamma^{i_2}_{(2)k}). \tag{30}$$

The result (29) can conveniently be represented in terms of the absolute differential as follows:

$$D[F_{(1)}(X) \cdot F_{(2)}(X)] = [DF_{(1)}(X)] \cdot F_{(2)}(X) + F_{(1)}(X) \cdot [DF_{(2)}(X)] \tag{31}$$

or

$$D[F_{(1)}(X) \cdot F_{(2)}(X)] = [F'_{(1)}(X) \cdot F_{(2)}(X) + F_{(1)}(X) \cdot F'_{(2)}(X)] \cdot dX. \tag{32}$$

From the last formula we obtain the relation

$$[F_{(1)}(X) \cdot F_{(2)}(X)]' = F'_{(1)}(X) \cdot F_{(2)}(X) + F_{(1)}(X) \cdot F'_{(2)}(X). \tag{33}$$

It remains to clarify whether the transformation law of the objects $\gamma^i_{(3)k}$ under the transitions to arbitrary coordinate system is correct. With this aim the formula (30) should be written in a varied form:

$$\gamma^i_{(3)k} = p^i_{i_1 i_2} \cdot (\gamma^{i_1}_{(1)k} f^{i_2}_{(2)} + f^{i_1}_{(1)} \gamma^{i_2}_{(2)k}) + (\Gamma^i_{km} p^m_{i_1 i_2} - \Gamma^m_{ki_1} p^i_{mi_2} - \Gamma^m_{ki_2} p^i_{i_1 m}) \cdot f^{i_1}_{(1)} f^{i_2}_{(2)}, \tag{34}$$

where $\Gamma^j_{im} \equiv 0$ with the respect to our special coordinate system; however, under the transition to an arbitrary coordinate system the objects Γ^j_{ik} transform like ordinary connection objects and in general $\Gamma^{j'}_{i'k'} \neq 0$. The condition $\Gamma^j_{ik} \equiv 0$ can also be replaced to apply the more general condition

$$\Gamma^i_{km} p^m_{i_1 i_2} - \Gamma^m_{ki_1} p^i_{mi_2} - \Gamma^m_{ki_2} p^i_{i_1 m} \equiv 0 \tag{35}$$

and, moreover, the three coefficients Γ in (35) can be regarded as different. It is possible to restrict ourselves to but the class of generalized-analytical function obeying the property

$$({}^{(1)}\Gamma^i_{km} p^m_{i_1 i_2} - {}^{(2)}\Gamma^m_{ki_1} p^i_{mi_2} - {}^{(3)}\Gamma^m_{ki_2} p^i_{i_1 m}) \cdot f^{i_1}_{(1)} f^{i_2}_{(2)} \equiv 0. \tag{36}$$

Given the special coordinate system. If one has $\Gamma^i_{jk} \equiv {}^{(1)}\Gamma^i_{jk} \equiv {}^{(2)}\Gamma^i_{jk} \equiv {}^{(3)}\Gamma^i_{jk} \equiv 0$, then the tensor p^k_{ij} is transported "parallel" without any changes in components.

Thus, the poly-product of two generalized-analytical functions of poly-number variable is again a generalized-analytical function, and the formula (33) takes place for derivatives if one adopts that the "connection coefficients" associated to the tensor p^k_{ij} with respect to the special coordinate system vanishes identically over all three indices. In terms of the pairs $\{f^i, \gamma^i_k\}$ the poly-product of two generalized-analytical function can be written as follows:

$$\{f^{i_1}_{(1)}, \gamma^{i_1}_{(1)}\} \cdot \{f^{i_2}_{(2)}, \gamma^{i_2}_{(2)}\} = \{p^i_{i_1 i_2} f^{i_1}_{(1)} f^{i_2}_{(2)}, p^i_{i_1 i_2} \cdot (\gamma^{i_1}_{(1)k} f^{i_2}_{(2)} + f^{i_1}_{(1)}) \gamma^{i_2}_{(2)k})\}. \quad (37)$$

So, the polynomial or the converged series with real or poly-number coefficients of one or several generalized-analytical functions is a generalized-analytical function. The ordinary differentiation rules are operative for the respective derivative (which was denoted my means of the prime (′)) of such polynomials and series, whenever the tensor p^k_{ij} with respect to the special coordinate system vanishes identically over all three indices.

Since in such a theory of generalized-analytical functions of poly-number variable (in which the "connection objects" as well as the gamma-objects are different for each tensor and, generally speaking, for each index), the concept of "parallel transportation" is deprived of the geometrical simplicity that is characteristic of the spaces of affine connection, the Riemannian and pseudo-Riemannian spaces included. This notwithstanding, the concepts of absolute differential and covariant derivative can readily be extended on the basis of invariance of their form with respect to any curvilinear coordinate system. The covariant derivative $\widetilde{\nabla}_k$ for arbitrary tensor is defined quite similarly to the way which is followed to define the covariant derivative ∇_k in the spaces of affine connection; at the same time, for each tensor and probably for each index there exist, in general, their own "connection objects" or gamma-objects. The respective differential is constructed in accordance with the definition

$$\widetilde{D} := dx^k \cdot \widetilde{\nabla}_k. \quad (38)$$

Here, the converted indices can not be ignored, for "connection coefficients" correspond to them.

The Cauchy-Riemann relations (24) are necessary and sufficient conditions in order that f^i be a generalized-analytical function. Let us show that these relations can be written in an explicitly invariant form if the matrix composed of the numbers

$$q_{ij} = p^r_{im} p^m_{rj}, \quad (39)$$

is non-singular, that is if

$$q = \det(q_{ij}) \neq 0. \quad (40)$$

In this case the inverse matrix (q_{ij}) forms the tensor (q^{ij}) showing the properties

$$q_{jk} q^{ki} = q^{ik} q_{kj} = \delta^i_j. \quad (41)$$

Whence, when the formula (22) is applied instead of the formulae (23) and (24), we get the invariant expression for the derivative

$$f'^i = q^{is} p^r_{sm} \widetilde{\nabla}_r \cdot f^m \quad (42)$$

and for the Cauchy-Riemann relations

$$\widetilde{\nabla}_k f^i - p^i_{kj} \cdot q^{js} p^r_{sm} \widetilde{\nabla}_r f^m = 0. \quad (43)$$

Let us turn to the generalized-analytical functions $F_{(1)}(X)$ and $F_{(2)}(X)$, which are constrained by the relation

$$F_{(2)}(X) = F(X) \cdot F_{(1)}(X), \quad (44)$$

where $F(X)$ – some function of poly-number variable. The function is generalized-analytical in the field where the function $F_{(1)}(X)$ is not a divisor of zero. In this case

$$F(X) = \frac{F_{(2)}(X)}{F_{(1)}(X)}, \tag{45}$$

$$\widetilde{D}F(X) = \frac{F_{(1)}(X)\widetilde{D}[F_{(2)}(X)] - \widetilde{D}[F_{(1)}(X)]F_{(2)}(X)}{F_{(1)}^2(X)} \tag{46}$$

or

$$F'(X) = \frac{F_{(1)}(X)F'_{(2)}(X) - F'_{(1)}(X)F_{(2)}(X)}{F_{(1)}^2(X)}. \tag{47}$$

If

$$F(X) = F_{(2)}[F_{(1)}(X)], \tag{48}$$

then the function $F(X)$ is generalized-analytical with

$$F'(X) = F'_{(2)}(F_{(1)}) \cdot F'_{(1)}(X). \tag{49}$$

3. Similar geometries and conformal transformations

Actually, we are interested in not only the pair $\{M_n, P_n\}$ and generalized-analytical functions $\{f^i, \gamma^i_k\}$ but (eventually) possible ways of application of these notions to constructing physical models and solving new physical proiblems. Two spaces in which congruences of extremals (geodesics) coincide are similar in many respects. The extremals are meant to be solutions to set of equations for definition of curves over which the length of the curve acquires its extremum; alternatively, they are meant to be the curves which in a given geometry are defined to be geodesics (for example, geodesics in geometries of affine connection). However, for some physical as well as mathematical problems it is not of great importance which length element is used in applied space, – a real use is made to only the set of equations that define extremals (or to extremals proper). We shall say that two n–dimensional geometries are *similar*, if there exist such coordinate systems and parameters along curves that with respect to them the equations for extremals are equivalent and the initial and/or final date set forth in one space may also be given in another space.

All the set of generalized-analytic functions can be broken into the subsets $\{f^i, \Gamma^k_{ij}\}$ that involve the same connection coefficients Γ^k_{ij}, so that for all generalized-analytic functions from the subset the relation

$$\Gamma^i_{kj} f^j = \gamma^i_k \tag{50}$$

is fulfilled. It should be stressed (once more) that the coefficients Γ^k_{ij} are independent of any choice of functions in the subset $\{f^i, \Gamma^k_{ij}\}$. Generally speaking, the subset may be formed by only one generalized-analytic function. If f^i and γ^i_k are prescribed, then the relations (50) can be treated to be a set of equations for definition of the coefficients Γ^k_{ij}. Having find and fixed them, they can be applied for all tensors and indices, thereafter we get a due possibility to work with the space of affine connection $L_n(\Gamma^k_{ij})$ in which the set of equations for geodesics is of the form

$$\frac{d^2x^i}{d\sigma^2} = -\Gamma^i_{kj}\frac{dx^k}{d\sigma}\frac{dx^j}{d\sigma}. \tag{51}$$

Generally speaking, in this way we loose the possibility to use the poly–number product for construction of new generalized–analytical functions and should give up the simple differentiation rules (33). In the last case the covariant derivative $\widetilde{\nabla}_k$ in the special coordinate system must act on the tensor p^i_{kj}. In order to have simultaneously on the subset $\{f^i, \Gamma^k_{ij}\}$ the poly-number product of generalized–analytical functions and the rules (33), which application yields again a generalized–analytical function, we are to restrict ourselves to the functions subjected to the condition (36) with $\Gamma^i_{jk} \equiv {}^{(1)}\Gamma^i_{jk} \equiv {}^{(2)}\Gamma^i_{jk} \equiv {}^{(3)}\Gamma^i_{jk}$.

Let us require that the space $L_n(\Gamma^i_{jk})$ be similar to a Riemannian or pseudo–Riemannian one $V_n(g_{ij})$, where g_{ij} is a (fundamental) metric tensor. Then instead of (50) we get the system of equations

$$\left[\frac{1}{2}g^{im}\left(\frac{\partial g_{km}}{\partial x^j}+\frac{\partial g_{jm}}{\partial x^k}-\frac{\partial g_{kj}}{\partial x^m}\right)+\frac{1}{2}(p_k\delta^i_j+p_j\delta^i_k)+S^i_{kj}\right]\cdot f^j=\gamma^i_k, \tag{52}$$

where S^i_{kj} stands for an arbitrary tensor (torsion tensor) obeying the property of skew–symmetry with respect to two indices, and p_i may be an arbitrary one–covariant tensor [1]. This system may be used to define the fundamental tensor g_{ij}.

There exist such Finslerian spaces which are not of Riemannian or pseudo–Riemannian type, but in which, however, one has the system of equations

$$\frac{d^2x^i}{d\sigma^2}=-\Gamma^i_{kj}[L(dx;x)]\cdot\frac{dx^k}{d\sigma}\frac{dx^j}{d\sigma}, \tag{53}$$

where the coefficients $\Gamma^i_{kj}[L(dx;x)]$ are defined by means of a relevant metric function $L(dx^1,\dots,dx^n;x^1\dots,x^n)$ of Finsler type. The corresponding Finsler spaces are similar to spaces of affine connection endowed with the connection coefficients Γ^i_{kj} deviated possibly from the coefficients $\Gamma^i_{kj}[L(dx;x)]$ by occurrence of an additive torsion tensor and/or an additive tensor $\frac{1}{2}(p_k\delta^i_j+p_j\delta^i_k)$ [1].

Let a generalized–analytical functions define spaces of the affine connection $L_n({}^{(1)}\Gamma^k_{ij})$ and $L_n({}^{(2)}\Gamma^k_{ij})$ similar to corresponding Riemannian or pseudo–Riemannian spaces $V_n(g_{ij})$ and $V_n(K^2_V g_{ij})$ and/or the Finslerian spaces $F_n[L(dx;x)]$ and $F_n[K_F L(dx,x)]$, where $K_V(x^1,..,.x^n)>0$, $K_F(x^1,...,x^n)>0$ – scalar functions (invariants). Then the transformation (coordinate and/or in the space of generalized–analytical functions) going over the set $f^i_{(1)}$ in the set $f^i_{(2)}$, can be called conformal, for under this procedure one has

$$g_{ij}(x)\to K^2_V(x)\cdot g_{ij} \tag{54}$$

and

$$(dx;x)\to K_F(x)\cdot L(dx;x). \tag{55}$$

4. Possible additional requirements

From the definition of a generalized–analytical function it follows that it is possible to present the function by choosing two arbitrary one–covariant fields $f^i(x^1,\dots,x^n)$ and $f'^i(x^1,\dots,x^n)$. Then the formula (23) entails the following representation for the gamma–objects:

$$\gamma^i_k=-\frac{\partial f^i}{\partial x^k}+p^i_{kj}f'^j \tag{56}$$

The Cauchy–Riemann conditions are fulfilled automatically. So, to get the field equations for the unknown function–components $f^i(x^1,\dots,x^n)$ and $f'^i(x^1,\dots,x^n)$, it is nesessary to set forth at least $2n$ additional relations, for example, some partial differential equations of the first–order with respect to $f^i(x^1,\ldots,x^n)$ and $f'^i(x^1,\ldots,x^n)$.

(1): Let us consider the subset of generalized–analytical functions f^i such that

$$\tilde{D}F(x)\equiv 0, \quad \leftrightarrow \quad \tilde{\nabla}_k f^i \equiv 0, \quad \leftrightarrow \quad f'^i \equiv 0 \tag{57}$$

In this case the Cauchy–Riemann conditions are fulfilled automatically and arbitrary vector–function coupled with $\gamma^i_k = -\frac{\partial f^i}{\partial x^k}$, that is the pair $\{f^i, -\frac{\partial f^i}{\partial x^k}\}$, is a generalized–analytical function. It is important to note that the properties of poly–numbers do not influence this procedure. In other words, this subset (treated on the level of the Cauchy–Riemann conditions) are independent of any choice of the system of poly–numbers.

(2): If instead of the conditions (57) we assume the relations

$$\tilde{D}F(X)=\lambda\cdot F(X)\cdot dX, \quad \leftrightarrow \quad \tilde{\nabla}_k f^i = \lambda\cdot p^i_{kj}\cdot f^j, \quad \leftrightarrow \quad f'^i=\lambda\cdot f^i, \tag{58}$$

where λ is a real number, then the pairs $\{f^i, -\dfrac{\partial f^i}{\partial x^k}+\lambda p^i_{kj}f^j\}$ with arbitrary vector–functions f^i will form the subset of the generalized-alanytical functions which to some extent account for properties of poly–numbers.

(3): Farther generalizing the requirements (57) and (58) can be formulated in the form

$$F'(X)=\Lambda\cdot F(X), \tag{59}$$

where

$$\Lambda=\lambda^1 e_1+\lambda^2 e_2+...+\lambda^n e_n \tag{60}$$

an arbitrary poly–number. In this case the pair

$$\left\{f^i, -\frac{\partial f^i}{\partial x^k}+p^i_{kj}p^j_{mr}\lambda^m f^j\right\} \tag{61}$$

will be the generalized-alanytical functions.

(4): Using the formulas (23) and (24), we can prove the following statement. If the relations

$$1)\ \Gamma^i_{kj}f^j=\gamma^i_k, \tag{62}$$

$$2)\ \Gamma^i_{1j}p^j_{kr}-p^i_{kj}\Gamma^j_{1r}=0, \tag{63}$$

$$3)\ \frac{\partial\Gamma^i_{1r}}{\partial x^k}-\frac{\partial\Gamma^i_{kr}}{\partial x^1}+\left[(\Gamma^i_{kj}-p^i_{km}\Gamma^m_{1j})\Gamma^j_{1r}-\Gamma^i_{1j}(\Gamma^j_{kr}-p^j_{km}\Gamma^m_{1r})\right]=0 \tag{64}$$

hold, then together with the generalized-alanytical pair $\{f^i,\gamma^i_k\}$, the pair

$$\{f'^i,\Gamma^i_{kj}f'^j\},\{f''^i,\Gamma^i_{kj}f''^j\},\ldots,\{f^{(m)i},\Gamma^i_{kj}f^{(m)j}\},\ldots \tag{65}$$

are also generalized-alanytical. In the last formulas the notation

$$f^{(m)i}\equiv\frac{\partial f^{(m-1)j}}{\partial x^1}+\Gamma^i_{1j}f^{(m-1)j} \tag{66}$$

has been used.

(5): One additional requirements can sound: for the subset $\{f^i,\Gamma^i_{kj}\}$ of generalized-alanytical functions a Riemannian or pseudo–Riemannian geometry $V_n(g_{ij})$ similar to the affine connection geometry $L_n(\Gamma^i_{jk})$ can be found.

(6): If a Finsler space $F_n[L(dx;x)]$ is similar to a space of affine connection, then one among possible requirements can claim that the subset $\{f^i, \Gamma^i_{jk}\}$ give rise to an affine connection geometry similar to the Finsler geometry $F_n[L(dx;x)]$.

(7): Let

$$x^i = x^i(\tau) \tag{67}$$

be a parametric presentation of some curve joining two points $x^i_{(0)} = x^i(0)$, $x^i_{(0)} = x^i(0)$, that is, the parameter along curves varies in the limits $\tau \in [0;1]$. Let us consider the functional with integration along indicated curve

$$\begin{aligned} I[x^i(\tau)] &= \int_0^1 F(X)\, dX = \left[\int_0^1 p^i_{kj} f^k(x^1(\tau), \ldots, x^n(\tau)) dx^j\right] \cdot e_i \\ &= \left[\int_0^1 p^i_{kj} f^k \frac{dx^j}{d\tau}\right] \cdot e_i, \end{aligned} \tag{68}$$

where $F(X)$ – some generalized-alanytical function, and require that value of the integral (68) be independent of integration way, in which case the variation of this functional at fixed ends of curves should vanish, that is the Euler conditions

$$\frac{d}{d\tau}\left(p^i_{kj} f^j\right) - p^i_{mj} \frac{\partial f^j}{\partial x^k} \frac{dx^m}{d\tau} = 0 \tag{69}$$

or

$$\left(p^i_{kj} \frac{\partial f^j}{\partial x^m} - p^i_{mj} \frac{\partial f^j}{\partial x^k}\right) \cdot \frac{dx^m}{d\tau} = 0 \tag{70}$$

must be valid. Assuming that $x^i(\tau)$ are arbitrary smooth functions, from these equations we get

$$p^i_{kj} \frac{\partial f^j}{\partial x^m} - p^i_{mj} \frac{\partial f^j}{\partial x^k} = 0, \tag{71}$$

or, recollecting that $\{f^i, \gamma^i_k\}$ is a generalized–analytic pair,

$$p^i_{kj} \gamma^i_m - p^i_{mj} \gamma^j_k = 0. \tag{72}$$

From these relations it ensues that for the functions f^i the Cauchy–Riemann conditions (11) hold fine.

Thus, the assumption of independence of the integral (68) of the path leads to the conclusion that the function $F(X)$ is analytical, that is such an assumption is superfluous for non–trivial generalization of the concept of analyticity.

5. Case H_4

It is convenient to work with the associative-commutative hypercomplex numbers in term of the ψ-basis which relates to the basis

$$e_1 = 1, e_2 = j, e_3 = k, e_4 = jk, \qquad j^2 = k^2 = (jk)^2 = 1 \tag{73}$$

by means of the linear dependence

$$e_i = s^j_i \cdot \psi_j, \tag{74}$$

where

$$s_i^j = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}, \qquad s_i^k \cdot s_k^j = 4 \cdot \delta_i^j. \tag{75}$$

For the basis elements $\psi_1, \psi_2, \psi_3, \psi_4$ the multiplication law

$$\psi_i \cdot \psi_j = p_{ij}^{(\psi)k} \cdot \psi_k \tag{76}$$

involves the characteristic numbers

$$p_{ij}^{(\psi)k} = \begin{cases} 1, & \text{if } i = j = k, \\ 0, & \text{in other cases} \end{cases} \tag{77}$$

We shall use the following notation:

$$X = x^1 e_1 + ... + x^4 e_4 = \xi^1 \psi_1 + ... + \xi^4 \psi_4 \tag{78}$$

and

$$F(X) = \varphi^1(\xi^1, ..., \xi^4) \cdot \psi_1 + \varphi^4(\xi^1, ..., \xi^4) \cdot \psi_4. \tag{79}$$

Thus, if $\varphi^i(\xi^1, ..., \xi^4)$ – a generalized-analytical function of the H_4-variable used, then such a vector-function $\varphi'^i(\xi^1, ..., \xi^4)$ can be found that

$$\frac{\partial \varphi^i}{\partial \xi^k} + \gamma_k^{(\psi)i} = p_{kj}^{(\psi)i} \cdot \varphi'^j. \tag{80}$$

Taking into account (77), we get

$$\left.\begin{array}{llll}
\dfrac{\partial \varphi^1}{\partial \xi^1} + \gamma_1^{(\psi)1} = \varphi'^1, & \dfrac{\partial \varphi^1}{\partial \xi^2} + \gamma_2^{(\psi)1} = 0, & \dfrac{\partial \varphi^1}{\partial \xi^3} + \gamma_3^{(\psi)1} = 0, & \dfrac{\partial \varphi^1}{\partial \xi^4} + \gamma_4^{(\psi)1} = 0, \\
\dfrac{\partial \varphi^2}{\partial \xi^1} + \gamma_1^{(\psi)2} = 0, & \dfrac{\partial \varphi^2}{\partial \xi^2} + \gamma_2^{(\psi)2} = \varphi'^2, & \dfrac{\partial \varphi^2}{\partial \xi^3} + \gamma_3^{(\psi)2} = 0, & \dfrac{\partial \varphi^2}{\partial \xi^4} + \gamma_4^{(\psi)2} = 0, \\
\dfrac{\partial \varphi^3}{\partial \xi^1} + \gamma_1^{(\psi)3} = 0, & \dfrac{\partial \varphi^3}{\partial \xi^2} + \gamma_2^{(\psi)3} = 0, & \dfrac{\partial \varphi^3}{\partial \xi^3} + \gamma_3^{(\psi)3} = \varphi'^3, & \dfrac{\partial \varphi^3}{\partial \xi^4} + \gamma_4^{(\psi)3} = 0, \\
\dfrac{\partial \varphi^4}{\partial \xi^1} + \gamma_1^{(\psi)4} = 0, & \dfrac{\partial \varphi^4}{\partial \xi^2} + \gamma_2^{(\psi)4} = 0, & \dfrac{\partial \varphi^4}{\partial \xi^3} + \gamma_3^{(\psi)4} = 0, & \dfrac{\partial \varphi^4}{\partial \xi^4} + \gamma_4^{(\psi)4} = \varphi'^4.
\end{array}\right\} \tag{81}$$

These relations involve the expression for the derivative

$$\varphi'^i = \frac{\partial \varphi^i}{\partial \xi^{i_-}} + \gamma_{i_-}^{(\psi)i} \tag{82}$$

($i = i_-$, for which no summation is assumed), and also the Cauchy-Riemann relations

$$\frac{\partial \varphi^i}{\partial \xi^k} + \gamma_k^{(\psi)i} = 0, \ i \neq k. \tag{83}$$

The space H_4 is the metric (Finslerian) space in which the length element ds is expressible through the form $d\xi^1 d\xi^2 d\xi^3 d\xi^4$ in a conic region defined possibly in various ways. Let us stipulate that

$$ds = \sqrt[4]{d\xi^1 d\xi^2 d\xi^3 d\xi^4}, \tag{84}$$

assuming that the region is prescribed by the inequalities

$$d\xi^1 \geq 0,\ d\xi^2 \geq 0,\ d\xi^3 \geq 0,\ d\xi^4 \geq 0\ . \tag{85}$$

Let us consider the four-dimensional Finslerian geometry with the length element of the form

$$ds = \sqrt[4]{\kappa^4 \cdot d\xi^1 d\xi^2 d\xi^3 d\xi^4}, \tag{86}$$

where $\kappa \equiv \kappa(d\xi^1 d\xi^2 d\xi^3 d\xi^4) > 0$. Such a geometry is not Riemannian or pseudo-Riemannian. Let us show that such a geometry is similar (according to terminology adopted above) to some affine geometry with a connection $L_4(\Gamma^i_{kj})$. Let us write equations for extremals of this Finslerian space by using the tangential equation of indicatrix [2]:

$$\Phi(p_1, ..., p_4; \xi^1, ..., \xi^4) = 0\ , \tag{87}$$

where

$$\Phi(p;\xi) = p_1 p_2 p_3 p_4 - \left(\frac{\kappa}{4}\right)^4\ , \tag{88}$$

and

$$p_i = \frac{\partial(ds)}{\partial(d\xi^i)} = \frac{1}{4} \cdot \frac{\sqrt[4]{\kappa^4 \cdot d\xi_1 d\xi_2 d\xi_3 d\xi_4}}{d\xi^i}. \tag{89}$$

Then the set of equations for definition of extremals reads

$$\left.\begin{array}{c} \dfrac{d\xi^1}{\frac{\partial\Phi}{\partial p_1}} = ... = \dfrac{d\xi^4}{\frac{\partial\Phi}{\partial p_4}} = \dfrac{dp_1}{-\frac{\partial\Phi}{\partial\xi^1}} = ... = \dfrac{dp_4}{-\frac{\partial\Phi}{\partial\xi^4}}, \\ \Phi(p,\xi) = 0; \end{array}\right\} \tag{90}$$

or

$$d\xi^i = \frac{\partial\Phi}{\partial p_i} \cdot \lambda \cdot d\tau,\ dp_i = -\frac{\partial\Phi}{\partial\xi^i} \cdot \lambda \cdot d\tau,\ \ \Phi(p;\xi) = 0, \tag{91}$$

where τ – a parameter along extremals, and $\lambda \equiv \lambda(p;\xi) \neq 0$ – a function. For the tangential equation of the indicatrix (87), (88) the set of equations (91) takes on the form

$$\dot{\xi^i} = \frac{p_1 p_2 p_3 p_4}{p_i} \cdot \lambda, \quad \dot{p^i} = (\frac{1}{4})^4 \frac{\partial k^4}{\xi^i} \cdot \lambda, \quad p_1 p_2 p_3 p_4 = \left(\frac{k}{4}\right)^4, \tag{92}$$

with

$$\dot{\xi^i} = \frac{d\xi^i}{d\tau},\ \ \dot{p_i} = \frac{dp_i}{d\tau}. \tag{93}$$

Let us consider $\lambda = \lambda(\xi) > 0$ to be a function of only coordinates. Then, by explicating p_i, we get the set of equations for definition of extremals in the Finslerian space (86) in the form

$$\ddot{\xi^i} = -\Gamma^i_{kj}\ \dot{\xi^k}\dot{\xi^j}, \tag{94}$$

where

$$\Gamma^i_{kj} = -\left\{\begin{array}{ll} \dfrac{\partial ln\left(\dfrac{\lambda}{\lambda_0}\right)}{\partial\xi^j}, & \text{if } i=j=k, \\ \delta^i_k \dfrac{\partial ln\left(\dfrac{\sigma}{\sigma_0}\right)}{\partial\xi^j}, & \text{in other cases;} \end{array}\right. \tag{95}$$

$$\sigma = \left(\frac{\kappa}{4}\right)^4 \cdot \lambda, \tag{96}$$

λ_0 and σ_0 are constants of relevant dimensions. Let us write down explicitly the coefficients Γ^i_{kj}:

$$(\Gamma^1_{kj}) = -\begin{pmatrix} \dfrac{\partial ln\left(\dfrac{\lambda}{\lambda_0}\right)}{\partial\xi^1} & \dfrac{\partial ln\left(\dfrac{\sigma}{\sigma_0}\right)}{\partial\xi^2} & \dfrac{\partial ln\left(\dfrac{\sigma}{\sigma_0}\right)}{\partial\xi^3} & \dfrac{\partial ln\left(\dfrac{\sigma}{\sigma_0}\right)}{\partial\xi^4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{97}$$

$$(\Gamma^2_{kj}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \dfrac{\partial ln\left(\dfrac{\sigma}{\sigma_0}\right)}{\partial\xi^1} & \dfrac{\partial ln\left(\dfrac{\lambda}{\lambda_0}\right)}{\partial\xi^2} & \dfrac{\partial ln\left(\dfrac{\sigma}{\sigma_0}\right)}{\partial\xi^3} & \dfrac{\partial ln\left(\dfrac{\sigma}{\sigma_0}\right)}{\partial\xi^4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{98}$$

$$(\Gamma^3_{kj}) = -\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \dfrac{\partial ln\left(\dfrac{\sigma}{\sigma_0}\right)}{\partial\xi^1} & \dfrac{\partial ln\left(\dfrac{\sigma}{\sigma_0}\right)}{\partial\xi^2} & \dfrac{\partial ln\left(\dfrac{\lambda}{\lambda_0}\right)}{\partial\xi^3} & \dfrac{\partial ln\left(\dfrac{\sigma}{\sigma_0}\right)}{\partial\xi^4} \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{99}$$

$$(\Gamma^4_{kj}) = -\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \dfrac{\partial ln\left(\dfrac{\sigma}{\sigma_0}\right)}{\partial\xi^1} & \dfrac{\partial ln\left(\dfrac{\sigma}{\sigma_0}\right)}{\partial\xi^2} & \dfrac{\partial ln\left(\dfrac{\sigma}{\sigma_0}\right)}{\partial\xi^3} & \dfrac{\partial ln\left(\dfrac{\lambda}{\lambda_0}\right)}{\partial\xi^4} \end{pmatrix}. \tag{100}$$

It will be noted that instead of the matrices (97) – (100) one can take their transforms. Thus, the Finslerian geometry with the length element (86) is similar to the geometry of the affine connection $L_4[\Gamma^i_{kj} + S^i_{kj} + \frac{1}{2}(p_k\delta^i_j + p_j\delta^i_k)]$, where S^i_{kj} – a tensor which is assumed to be skew-symmetric with respect to the subscripts, and p_k stands for an arbitrary one-covariant tensor.

Let us consider the generalized-analytical functions φ^i of H_4-variable that obey the additional condition 3), that is the pair

$$\left\{\varphi^i, -\frac{\partial\varphi^i}{\partial\xi^k} + p^{(\psi)i}_{kj}\mu^j\varphi^j\right\}, \tag{101}$$

where

$$\Lambda = \lambda^i \cdot e_i = \mu^j \cdot \psi_j. \tag{102}$$

Let us select from such pairs a subset $\{\varphi^i, \Gamma^i_{kj}\}$, where Γ^i_{kj} are defined by the matrices transposed to the matrices (97) – (100). In this way, the requirement 6) is retained. Then the pair (101) should fulfill the 16 relations (50) the first four of which are

$$\frac{\partial\varphi^1}{\partial\xi^1} = \mu^1\varphi^1 + \frac{\partial\, ln\left(\dfrac{\lambda}{\lambda_0}\right)}{\partial\xi^1}\varphi^1, \qquad \frac{\partial\varphi^1}{\partial\xi^2} = \frac{\partial\, ln\left(\dfrac{\sigma}{\sigma_0}\right)}{\partial\xi^2}\varphi^1, \tag{103}$$

$$\frac{\partial\varphi^1}{\partial\xi^3} = \frac{\partial\, ln\left(\dfrac{\sigma}{\sigma_0}\right)}{\partial\xi^3}\varphi^1, \qquad \frac{\partial\varphi^1}{\partial\xi^4} = \frac{\partial\, ln\left(\dfrac{\sigma}{\sigma_0}\right)}{\partial\xi^4}\varphi^1.$$

For the compatibility it is necessary and sufficient that the mixed derivatives obtained with the help of the formulae (103) be equal. A part of these equations, except for three ones, is automatically satisfied. If we consider all the 16 equations, not confining ourselves to the first four equations, we get the following 12 conditions:

$$\frac{\partial^2 ln\left(\dfrac{\kappa}{\kappa_0}\right)^4}{\partial\xi^i\xi^j} = 0, \; i \neq j; \tag{104}$$

from which it ensues that

$$ln\left(\frac{\kappa}{\kappa_0}\right)^4 = a_1(\xi^1) + a_2(\xi^2) + a_3(\xi^3) + a_4(\xi^4) \tag{105}$$

or

$$\kappa = \kappa_0 \cdot \exp\{[a_1(\xi^1) + a_2(\xi^2) + a_3(\xi^3) + a_4(\xi^4)]/4\}, \tag{106}$$

where a_i are four arbitrary functions of one real argument. Then from equations (103) and relevant equations for other components of the generalized-analytical function, we get

$$\varphi^i = \varphi^i_{(0)}\left(\frac{\kappa}{\kappa_0}\right)^4\left(\frac{\lambda}{\lambda_0}\right) b_i(\xi^{i-}) \cdot exp(\mu^{i-}\xi^i), \tag{107}$$

where

$$a_i(\xi^{i-}) = ln\left|b_i(\xi^{i-})\right|. \tag{108}$$

Thus, despite of two additional requirement, the generalized-analytical function (107) in general case is not reducible to an analytical function of H_4-variable, and besides we obtain the expression (106) for the coefficients κ in the metric function of the Finslerian space with the length element (86). If $\frac{\lambda}{\lambda_0} = \left(\frac{\kappa_0}{\kappa}\right)^4$, then φ^i is an analytical function.

If

$$\kappa = \kappa_0 \cdot \exp\{[(\xi^1)^2 + (\xi^2)^2 + (\xi^3)^2 + (\xi^4)^2]/4\}, \tag{109}$$

then with respect to the coordinates x^i

$$\kappa = \kappa_0 \cdot exp\{(x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2\}. \tag{110}$$

Conclusion

Having introduced the concept of the generalized-analytical function of poly-number variable in the present work, we have made the first step in the direction of constructing a relevant theory aiming to develop theoretical-physical models. An important ingredient of such investigations must be search for additional requirements to be obeyed by the generalized-analytical functions and for the consequences implied by the requirements. The conditions that lead to trivial results — to analytical functions — should especially be analyzed. This may admit formulating the properties that are forbidden to attribute proper generalized-analytical functions of poly-number variable (in contrast to analytical functions proper). As it has been shown above, the independence of integral of integration path relates to such properties. Of course, it is necessary to carry out a particular attentive study to compare the properties of analytical functions of complex variable and generalized-analytical functions of poly-number variable in case of the dimension exceeding 2. It can be hoped, therefore, that the concepts and results of the present work may face future novel theoretical-physical applications.

References

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[2] P. K. Rashevsky: *The Geometrical theory of partial differential equations*, M.-L. 1947 (in Russian).