CHRONOMETRY OF THREE-DIMENSIONAL TIME

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The concept of the multi-dimensional time has tried not once to take its place in natural science, but every time under the pressure of some paradox was rejected. Meanwhile a philosophical question: why the space admits quite a number of dimensions and the time dos not, still preserves. In this work a new attempt has been made to resolve the matter, by switching from the traditional quadratic metrics to the Finslerian one, which may admit an arbitrary degree of the vector component that is included into the metric function. Though the offered method enables us to build continuums of time of any natural dimensionality, in order to demonstrate the specificity of the raised topic this study will focus on a simple (after rather trivial two-dimensional case) example of three temporial dimensions.

1. Introduction

The idea of space is accepted much easier and vividly than the idea of time. This circumstance is conditioned by the fact that the space is looked over all at one time, and above all in the three-dimensional shape, meanwhile we see just a side of the time and only in one dimension. This situation forced some scientists "to get rid" of the time, either limiting to fixed problems or driving the time into the condition of an extra space dimension. The first approach is related to Archimedes, the latter approach for the first time appeared in the works of Galiley, reached perfection in Lagrange's and in fact reigns nowadays, – though The Special Theory of Relativity practically confronted the category of time to space, denoting them absolutely different in their essence, having differences already on the geometrical level.

There grows the belief formulated for the first time by Synge [1] that Euclid put the natural science on the wrong track, as he took the space but not the time as the fundamental idea of the science. The lack of any adopted term for time studying according to Synge is the proof of such disregard. He suggested that we should use the word "chronometry" to define the branch of science that deals with the idea of time in the same wide meaning as geometry does with the idea of space. Though Synge is unlikely to mean the multi-dimensional time, his statement is applicable to this aspect of the problem.

2. Two-dimensional time

The essence of the multy-dimensional time, that serves as an alternative to the multy-dimensional space, can be illustrated by a paradoxical-seeming statement: practically all physicians know about the two-dimensional time, but by tradition go on looking at it in another way. We mean the pseudo-Euclidean plane. It is surprising that among all the Euclidean spaces only the two-dimensional is distinguished with its unique peculiarities, it is worth mentioning the following.

Firstly, the theorem of Louiville, that enumerates the types of possible conformal transformations, coming to translations, rotations, dilatations and inversions, is true for all the pseudo-Euclidean spaces with 3 or more dimensions. In the two-dimensional case the list of their conformal transformations is by far longer.

Secondly, there are several concepts of the total product of the plane vectors, and the majority of them have the inverse ones; meanwhile in other pseudo-Euclidean spaces only scalar product is introduced, as well as division is not defined at all.

Thirdly, isotropic vectors always divide the pseudo-Euclidean planes with the signature $(1, n-1)$ into 3 simply connected domains, with an exception of the plane, with 4 such domains.

Fourthly, it does not matter which of the two typical coordinates of the Euclidean space we will choose as the temporal and which as the spatial, as the result will change to permutation. Another case appears in planes with a bigger number of dimensions, where such symmetry collapses and to the time we can apply only change of the sign.

And finally, only the plane admits the accordance with the associative-commutative algebra, whose main objects are called the double numbers. Their algebra has all the characteristics of usual algebras of real and complex numbers, including the product commutativity, with an exception of presence of specific objects, called the divisiors of zero. Each divisor of zero has a counterpart such that their product is a divisor. Though the double numbers are trivial in comparison with the complex, even such algebras cannot be related with pseudo-Euclidean spaces with more than 2 dimensions.

But, thinking that the uniform order starts with 3 and more dimensions, scientists, due to some reasons, don't notice or at best attribute it to the reducible nature of the two-dimensional space. It is interesting to note that we face practically the same in the Euclidean case: the two-dimensional representatives stand separately out and are juxtaposed with the algebra of complex numbers.

We can make a supposition basing on only these two examples that because of some reasons the connection of some metric spaces with the commutative-associative algebra make them in a way distinguished and that is why the very algebras and the corresponding spaces deserve a special attention.

When we stated in the beginning that we there was no reason to treat the pseudo-Euclidean space as a special case of the multy-dimensional time, we based on the fact that in the space there is no objective reason for us to distinguish which of its directions can act as time and which not. Then we must admit that in such a space all non-isotropic directions are equal in rights. Their differentiation by physical meaning takes place only after subjectively choosing one quadrant as the field of future.

Note. The subjective choice is related mostly to the world line, an element of whose length is interpreted as the proper time of an observer, and the future region is defined as the consequence of the line direction.

Only after the given procedure the points of the facing quadrant automatically acquire the meaning of the past actions, and the points of the two side – become absolutely distant. But few things will change on the pseudo-Euclidean plane if we choose to use any other quadrant as the field of the future, as only all the others will trade places. With an exception of this inessential-seeming moment, any further construction in the pseudo-Euclidean plane does not differ from the construction in its usual interpretation as the time-space.

But a move to 3 and more dimensions leads to the fact that the difference between the pseudo-Euclidean space-time and the dimension-corresponding pure time becomes principal, and moreover if we think of the conceptual multy-dimensional time as of a possible geometrical alternative to the space of the Special Theory of Relativity, it is important to revise not only mathematical, but also philosophical attitudes towards the structure of physical reality.

3. Three-dimensional time

To make a move from the two-dimensional time model to the three-dimensional let us use the observation that in the case of the pseudo-Euclidean plane the corresponding geometry becomes related with the idea of the commutative-associative hyper-complex number, which are related to the commutative-associative hypercomlex algebras. William Hamilton is the pioneer of hyper-complex numbers; while speaking at one of the sittings of the Royal Irish Academy he stated that if there existed geometry - the pure mathematical space science, there must be the same pure time science, and such a science should be algebra [2]. It is paradoxical but he on the example of the quaternions, discovered by himself, disproved the multitude of principally different algebras. But let us take his statement, as a presentiment of the great mathematician, and by analogy with the algebra of binary numbers we will try to make the algebra of triple number, and try to correspond with them geometry, or using Synge's suggestion, the chronometry of the three-dimensional time.

The presence of the basis in binary numbers makes the expression for the second degree of the module to take an absolutely symmetrical form:

$$|\mathbf{X}|^2 = x'_1 x'_2, \tag{1}$$

It indirectly shows that there must be a basis for the numbers that admittedly can be an algebraic analog to the vectors of the three-dimensional time. In this basis the fourth degree of the module becomes connected with the next absolutely symmetrical form out of three components:

$$|\mathbf{X}|^3 = x'_1 x'_2 x'_3. \tag{2}$$

It is not difficult to make sure that the algebra of such numbers exists, it is commutative and associative, and is the direct sum of three real algebras that continues the tendency that started at the example of binary numbers, whose algebra becomes the direct sum of the two real. As is well known, the one-dimensional time can be compared with the real numbers themselves, that is another confirmation of the chosen algebraic way of searching for models of the multy-dimensional time.

The manifolds for which the differentials of the vector length are expressed by means of the types (1) – (2), are well known in geometry and are called the Finslerian spaces with the Berwald-Moore metric function [3]. Usually under the term Finslerian spaces we understand the manifold of the most common type with a null meaning of curvature and torsion. The concerned metric (2) is defines the linear space, that is why it is in near relation with Euclidean and pseudo-Euclidean spaces, though they do not look alike in everything.

Let us call the linear Finslerian spaces, whose metric function in one of the bases looks like:

$$F(x') = \left|\prod_{i=1}^{n} x'_i\right|^{1/n}, \tag{3}$$

the *n-dimensional time*. To have not only axiomatical but also physical right to use this name let us interpret every point of the spaces as an event, and every line as a world line of an inertial reference frame.

Notes. The concept of an event is introduced in this way that though having something common with the classical analogue introduced by Minkowski, still differs from the latter. This is related to the fact that the concept of event in the multy-dimensional time stops having a single meaning and becomes dependent on the reference frame. In other words the same point of the space should be interpreted as different events if the world lines are separated by isotropic hypersurfaces. The concepts of time and space are as if substituting with one another. There are cases 2^n of such domains in n-dimensional time, and every point may have the same number of interpretations. But there does not emerge polysemy if we examine only the reference frames where the world lines lie only in the light cone, and the concept of event practically does not differ from its classical analogue.

In such reference frame the interval of proper time between an arbitrary pair of the equals the length of the vector related to the event. It follows from the symmetry of the examined spaces that all their non-isotropic directions are absolutely equal in rights if we decide to relate, according to the given above thesis, the length of the vector to the proper time in the distinguished reference frame then its justified to call the spaces, this time not by definition, rather than because of physical reasons, the multy-dimensional time.

But still preserves the question: whether such verities have any connection with the real world? To approach the answer let us try to examine the properties and peculiarities of the three-dimensional time. We will start from examining its structure and isotropic subspaces.

4. Light pyramids

The form (2) nullifies in the points that correspond to the three distinguished planes, defined by the equalization:

$$x_1' = 0, \quad x_2' = 0, \quad x_3' = 0. \tag{4}$$

The vectors lying on the plane have the zero meaning of the modulus and in this meaning are isotropic. At the same time, lines, that simultaneously belong to 2 planes (as well as the point of intersection of all the 3) automatically become marked out. As there are only three lines, it is quite natural to try to connect the vectors with the special basis. This basis is unique up to permutation and the form (2) given above defines the value of an arbitrary number module and also the length of the vector, – all being of the simplest shape. Concerning the originality of such basis, we will give it a proper name of the Absolute basis.

In this respect the concerned space turns out to be arranged in an absolutely another way, than the usual Euclidean and pseudo-Euclidean spaces, where there are no preferred bases (with an exception of the pseudo-Euclidean plane), and that is why we usually try to turn the studying of analogous geometries into a non-coordinate form. The existence of special bases in the multy-dimensional time means that if some day a connection between corresponding varieties and the physical reality will be found then some frame of reference will play a clearly distinguished role.

The isotropic planes (4) can be thought about for example as they are presented on Fig. 1. As we can see on the picture the three-dimensional space is divided by isotropic planes into 8 equal camer-oktants, that are domains of simple connectedness in fact. At the same time every camera is separated from the 3 side ones by the two-dimensional isotropic planes, it borders upon isotropic rays with another 3 cameras and with the opposite one it contacts through only one point. By analogy we can characterize, only taking into consideration the dimension, the mentioned above the two-dimensional time, where all the space is divided by isotropic lines into 4 camera-octants. Every quadrant is separated from 2 adjoining ones by isotropic rays, and with the opposite borders through a point. At the same time the one-dimension time also obeys the rule, as we can look upon

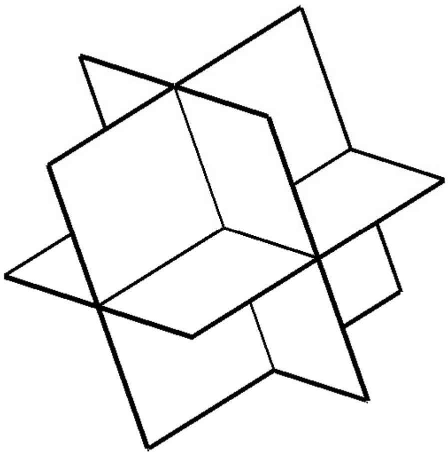

Figure 1: Isotropic planes of tree-dimensional time

the corresponding line as 2 opposite simply connected domains, divided by a special point, a zero that in a way can be considered to be an extreme singular case of the isotropic cone.

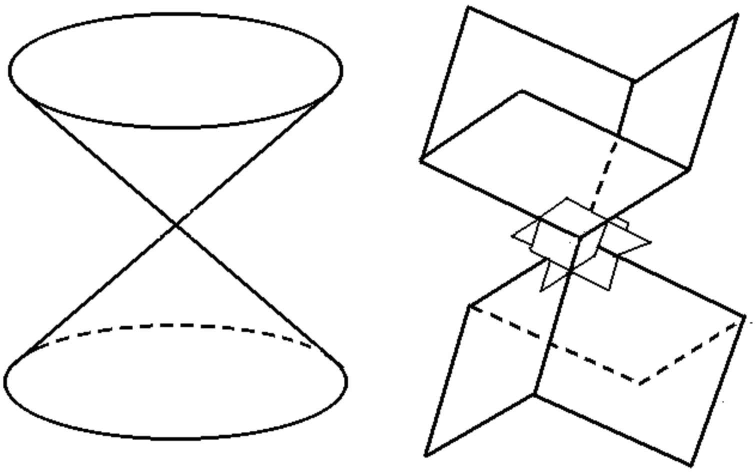

Figure 2: Light cones of tree-dimensional time (right) and tree-dimensional pseudo-Euclidian space (left)

If we choose 2 facing camera-octants from the 8 of the three-dimensional time and examine their united border we will get a figure depicted on Fig. 2. Such the sub-space looks like a light cone of the Euclidean space (depicted on the same picture to the left side) but for the fact that the first does not have a continuous axis symmetry. There are non-zero vectors in the inside of both facing octants, and the ends of the unit length vectors form 2 planes of a specific hyperboloid, which is the Finslerian analogue of the double-band hyperboloid of the pseudo-Euclidean space. Both figures are depicted on Fig. 3, the left corresponds to the three-dimensional time and represents only a quarter of the hyperboloid of space, which has 8 cavities, each for every simply connected area. The points of the figure satisfy the equalization: $|x_1'x_2'x_3'| = 1$, and its general form is represented on Fig. 4.

Among the unit vectors that are set against one and the same plane of such hyperboloid continuous transfers, exercised by the Abelian two-parameter group of linear

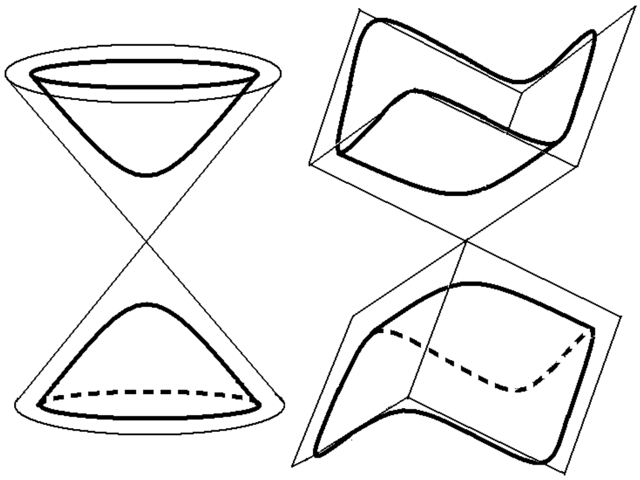

Figure 3: The fragments of unit hyperboloids

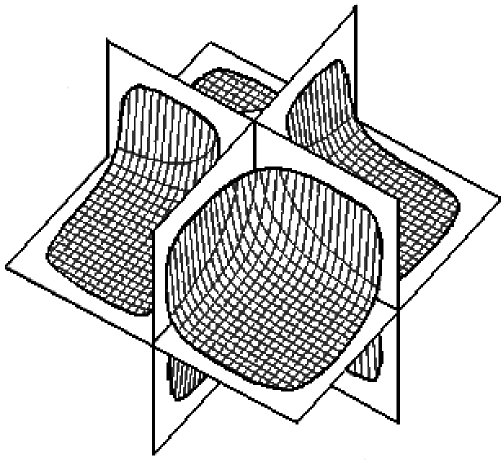

Figure 4: The eight-sheet hyperboloid of tree-dimensional time

transformations, is possible. The transformations can be displayed as a diagonal matrix:

$$\begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}, \tag{5}$$

with $a_1a_2a_3 = 1$. Transformations of the group are invariant to the interval of the three-dimensional time (2) and that is why it is its motion. In their character the motions are similar to the boosts of the corresponding pseudo-Euclidean space with the only difference that the points of the line stay static in the one-parameter turnings in space-time, and in the analogous case of the concerned space – only one single point. We will call transformations of the group the *hyperbolic turning of the three-dimensional time.*

Among motions of the space, apart from turnings, we can single out a three-parameter group of parallel shifts, that are a common idea in linear planes. There is no other continued transformation that would be invariant to the interval in the three-dimensional time.

The isotropic edges and unit hyperboloids of the distinguished group of facing octants whose ends are to end at infinity are depicted on Fig. 2 and Fig. 3, but due to

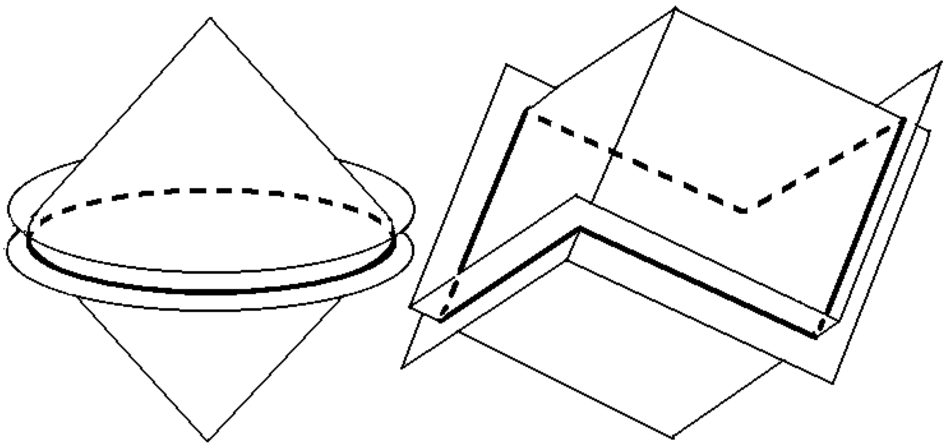

Figure 5: The two light cones couple intersection

the limited plane of the draft, their ends are cut short, but not at a plane, common for pseudo-Euclidean space, but in a more sophisticated way according to the following considerations. If we intersect the border of one of the octants with the border of the facing octant dislocated along their mutual axis we will get a rectilinear hexagon, and not a plane but the broken as it is demonstrated on Fig. 5. The volume that belongs to the interior of both octants is a common cube, and the mentioned above hexagon is composed of its edges that do not intersect the main axis.

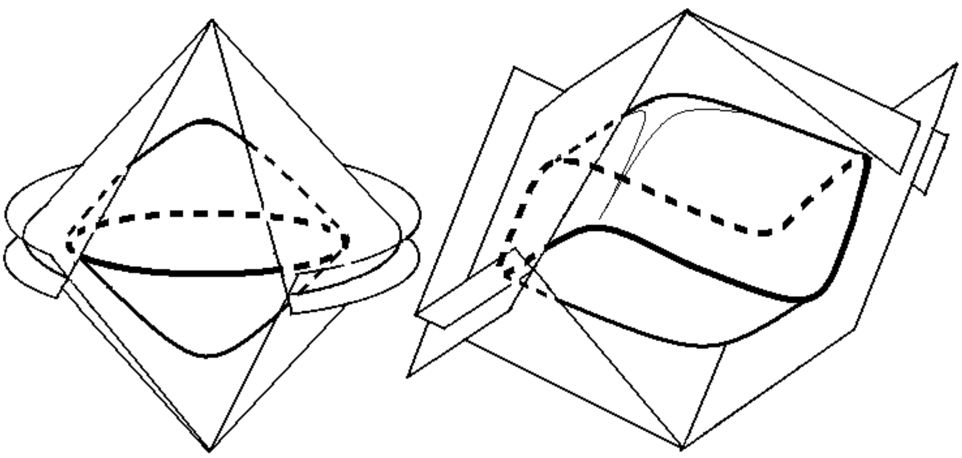

Figure 6: The two hyperboloids couple intersection with $0 < R < T$

Notes. We can say that in case of the n-dimensional time the figure that is the interception of two deposed towards each other facing cameras, consists of a half of $(n-2)$ edges of the formed by it hypercube, on top of all only edges that do not have common points with the main axis of symmetry participate in the formation.

If we construct two sets of concentric hyperboloids (per se they are Finslerian generalizing of spheres) inside the octants that form the cube with their centers in the opposite tops, the intersection of pairs with equal radius will result into a set of continuous closed graphs, whose form depends on the ratio of the corresponding to the curve radius of the hyperboloid R to half of the main diagonal of the cube T. When the radius of hyperboloids equal 0 they coincide with the isotropic edges of the octants, and their interception is a

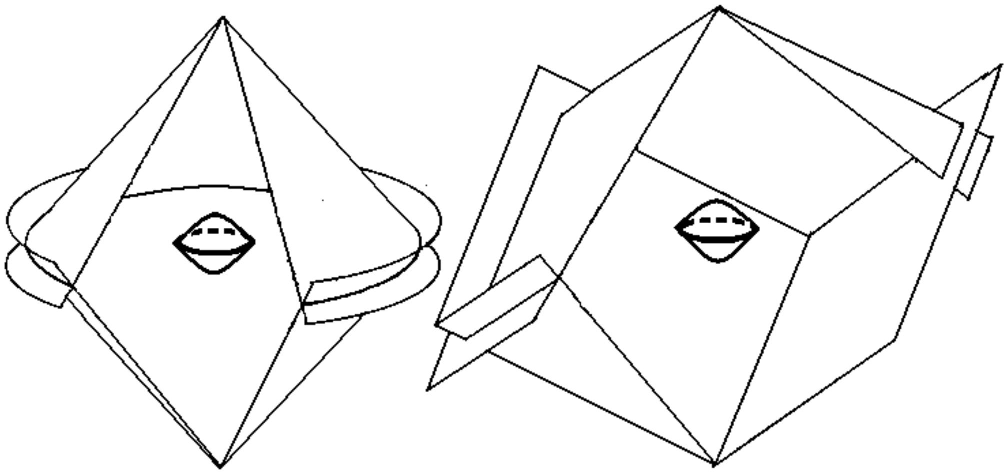

Figure 7: The two hyperboloids couple intersection with $R \approx T$

broken in space hexagon already examined on Fig. 5. When $0 < R < T$ the hyperboloids are intercepted on curves that look like the curve on Fig. 6. They are three-dimensional and have 6 round corners. While the value of the hyperboloid radius approaches to the value T the curves that are the result of their interception become more smooth and flattened out, and when $R \to T$ they turn into absolutely plane circumferences, though with infinitesimal radius Fig. 7.

In the three-dimensional pseudo-Euclidean space the analogous constructions lead to a group of concentric circumferences that lie in the same plane, you can see the circles on Fig. 5-7 to the right of them. The circumference that belongs to two light cones, that is corresponds to the interception of the pseudo-Euclidean sphere with $R = 0$ which in the Special Thery of Relativity is interpreted as a momentary position of the light front, that can be registered by the observer that is at the top of one of the cones, supposing that there is a flash at the top of the other. In general we should apply an analogous interpretation to the three-dimensional time case. So, the broken hexagon depicted on Fig. 5 can be interpreted as the multitude of points of the observer space, that is situated at the point T, with which it connects the momentary position of the light front, whose flash took place in $-T$. To make this situation true we must admit that the isotropic borders of the facing octants are analogues of the light cones of the past and future that corresponds in number of dimensions with the pseudo-Euclidean. This method looks rather natural and the only effort, in comparison with the common idea of the Special Theory of Relativity, we should make is to admit the borderness of the light cone. Taking into consideration that this borderness is executed in the space not available for the contemplation of the observer, the question whether is complies with the realities of our world turns out to be not so obvious.

Though we could save the name of light cones, usually used in the pseudo-Euclidean spaces in order not to emphasize peculiarities of geometry of the multy-dimensional time, for the isotropic borders of the simply-connected cameras, so let us call the corresponding figures the *light pyramids*, first of all singling out the pyramids of the past and future.

5. Planes of relative simultaneity

We should logically go further and accept an analogy not only between isotropic sub-spaces and the related to them light fronts but also we should put into correspondence with every common circle of two equal hyperboloids of the pseudo-Euclidean space an analogous curve, that is the interception of a pair of Finslerian spheres of the multy-dimensional time. There emerges quite a natural way to define the plane of the relative simultaneity of the three-dimensional time, as the same physical sense was played in the pseudo-Euclidean geometry by a plane represented with the above examined set of circles. Following the logic we should understand a multitude of points, equidistant in the meaning of the corresponding Finslerian metrics of two fixed points, under the simultaneous events of the multy-dimensional time. At the same time one of the fixed points coincides with the momentary position of the observer, and the second is the reflection of it with respect to the studies plenty of events.

The straight line that goes through the two points defines the inertial reference frame, but as it follows from the accepted definition of simultaneity now this property depends not only on the speed of the observer but also on his momentary position concerning the layer, to which he is going to give the equal time of performance. In the pseudo-Euclidean case (that has become practically classical) while defining the simultaneity meant only the relative speed of the relative speed of the reference frame, and the momentary position of the observer was not important. It is not so in the three-dimensional time and this circumstance seems to be one of the most important items, that differ the physical properties of the examined manifold from the common pseudo-Euclidean constructions.

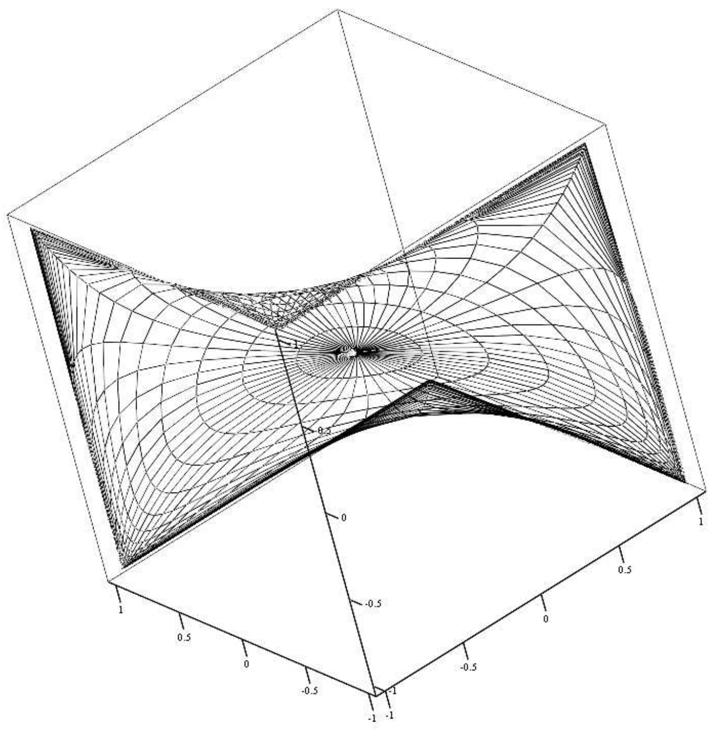

Figure 8: The simultaneous surface of three-dimensional time

It is convenient to describe the plane of simultaneity that corresponds to a fixed pair of points by an equalization that relates it coordinates to the coordinates of the initial affine space represented in the absolute basis. It is not difficult to get such equalization for an arbitrary pair of points, but it looks most vividly when momentary position of the observer is related to the point (T, T, T), and its reflexion has coordinates $(-T, -T, -T)$. In this case the equality of intervals leads to the equalization:

$$|(x'_1 + T)(x'_2 + T)(x'_3 + T)| = |(T - x'_1)(T - x'_2)(T - x'_3)|, \tag{6}$$

then after opening the brackets it leads to:

$$x'_1 x'_2 x'_3 + (x'_1 + x'_2 + x'_3)T^2 = 0. \tag{7}$$

The plane corresponding to the equalization is depicted on Fig. 8.

The curves examined on Fig. 5 and Fig. 7 mark points on the plane litarally equidistant from their geometrical center. Such curves in many ways are analogous to common concentric circles, though the related to it geometry does not coincides with the usual Euclidean.

On the other hand we can get a new group of curves, that corresponds to the multitude of radial lines of the Euclidean circle the canonic planes by intercepting the plane of simultaneity by canonic planes, called in the work [4] the *cones of rotation*, have tops in the point (T, T, T) and include the real axis. So, there is a net of curvilinear coordinates, that in the two-dimensional physical space play the same role as the polar scheme of coordinates does in the Euclidean plane.

Transformations that turn into themselves the plane of simultaneity so that the circles and radial curves at the same time map into the same curves and become in many ways analogous to spatial turns around the point of origin in the pseudo-Euclidean space, as the physical distance in either of the cases remain the same. But in the case of the three-dimensioal time these transformations are not linear, and on top of all do not leave invariant the three-dimensional intervals.

6. Physical distance and speed

It could seem that we have approached to the possibility of introduction into the three-dimensional time of two-dimensional physical distance and speed, it is enough to bring on the simultaneity plane in correspondence the set of circumferential and radial curves with the lines of the polar reference frame. But it is not like this. The fact is that the examined multitude does not admit the introduction as one-digit such physical notions as the distance and speed at least if the construction is based on the starting measurement of time intervals. What seems to be practically an obvious property of the pseudo-Euclidean spaces turns out to be not-compatible with the idea if the multy-dimensional time. This circumstance not only decr eases, but on the reverse increases the possibility of the multi-dimensional time to compete with the Minkowski space for being the geometrical basis of the real world. In fact, if we follow the idea of chronometry we should associate associate the time intervals, that are needed to send a desired signal and receive its reflection, with physical distance. But any attempt to unite this natural and vivid physical principle with the necessity of one-digitness comes upon obstacles. The idea of rejecting the one-digitness of the physical distance and speed seems to be a nice and far-reaching exit (cf. interpretations of quantum-mechanical uncertainty principle).

The above said does not mean that an entirely amorphous structure should replace the Euclidean geometry of the physical space. The analysis shows that our radical supposition touches upon not the quality, but only quantity aspect of the phenomenon. The distance and speed as independent physical categories are not completely excluded in the multy-dimensional time, but only change their status, getting the traits uncertainty on the initial geometric level. In particular the idea of equidistant in the physical meaning objects becomes dependent on which signals the observer, that defines this equidistance, uses as the reference. In its way the reference signals are defined by the principle of equality of proper times, where the hours pass in the corresponding inertial reference frames between sending, reflecting and receiving the signals. Taking into consideration that the time intervals are the only value that by definition are measurable in our Finslerian multitude, the task of distinguishing among the continuous specter of inclined world lines the ones would be characterized by the equality of intervals is quite possible. Let us note that we already used the method above, while defining the relatively simultaneous events. So, we can consider the signals to be etalons if their world lines start in one point, reach the plane of simultaneity and after refraction gather together and in another fixed point of the world line of the same observer. It is clear that all the intervals should be equal either before or after the refraction.

Such logic in constructing drives us to the fact that the physical space of the observer with its geometrical properties becomes in a way dependent on which set of reference signals define the geometry. So if the world lines of reference frames are practically parallel to the line of the observer, he starts to see a space, which in its characteristics practically coincides with the Euclidean. This is related to the fact that the ends of the vectors with the same value of the intervals in this cases lie (as it has been said above) on practically plane and ideal circle, and the latter while constructing the physical space plays the role of the Finslerian indicatrix. A common circle is the indicatrix of the two-dimensional Euclidean space. When tuning to the signals whose world lines are inclined more significantly, the ends of the corresponding vectors form this time not a circle, but a more sophisticated closed curve, which is not a plane one. At limit of the signals, whose speeds are interpreted as the light, this curve transforms into a broken hexagon, examined on Fig. 5. The geometry of the two-dimensional physical space is the Finslerian, and it is this geometry that differs greatly from the Euclidean, but in connection with the fact that the indicatrix even in this limit case is still closed and flattened out. The differences between the two geometries are not significant, in connection with which it is probably possible to mix them up, especially if the experimental cases are limited to low speeds.

So, if we suppose that our real world has a direct connection with the examined Finslerian geometry, the appearance of Euclidean and pseudo-Euclidean ideas in observer outlook should be a natural process of consistent approaches to a more exact description. On the other hand in our everyday life we use signals whose speed is by far lower than the light when we try to find the zones that manage the world. As the matter of fact we use the light only to identify the objects, and the distance is defined by other slower means – for example by a ruler. This circumstance leads us to the fact that when in special experiments really high-speed signals become of great importance, the geometry is considered to be defined before hand, and that is why even abnormal results will be treated anyhow, but only not in the direction of revising the obvious geometrical properties.

7. Conclusion

Among all the above listed properties and peculiarities of the three-dimensional time, as a representative of a very specific class (the non-linear) of Finslerian spaces, we should treat as the most important the one, thanks to which it is related to the most fundamental notion of mathematics – the number – which is the object of algebra, that has the most common arithmetical properties. We should emphasize ones more the fact that neither Euclidean nor pseudo-Euclidean spaces with three or more dimensions do not possess the analogous qualities. The quaternions and biquaternions used in similar situations are not genuine numbers, as there algebra has commutative multiplication, as the result of which the construction of a valuable theory that would generalize the theory of functions of the complex variable is not possible (or is extremely difficult). At the same time the given above examples demonstrate how common Euclidean and pseudo-Euclidean conceptions can come out of the idea of substitution of the pseudo-Euclidean metric to multy-temporal case – rather interesting and actual.

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