MAXWELL ELECTROMAGNETIC EQUATIONS IN THE UNIFORM MEDIUM. AN ALTERNATIVE TO THE MINKOWSKI THEORY OF SPECIAL RELATIVITY.

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Two known, alternative to each other, forms of presenting the Maxwell electromagnetic equations in a moving uniform medium are discussed. The commonly used Minkowski approach is based on two tensors; the relationships between them change their form under Lorentz transformations and take the shape of Minkowski equations, depending upon the 4-velocity of the moving particle in an inertial reference frame. In this approach, the wave equation for the electromagnetic 4-potential has a form which explicitly involves this 4-velocity vector of the reference frame. Hence, the Minkowski electrodynamics implies the absolute nature of mechanical motion.
An alternative formalism (proposed by Rosen & al.) may be constructed in new variables, when the Maxwell equations are written in terms of a single tensor. This form of Maxwell equations exhibits symmetry under modified Lorentz transformations in which, everywhere, instead of the vacuum speed of light c one uses the medium speed of light $c' < c$. Due to this symmetry, the formulation of Maxwell theory in this medium can be considered as invariant under the mechanical motion of the reference frame, while the transition must follow modified Lorentz formulas. The transition of the Maxwell equations to 4-potential leads to a simple wave equation which does not contain any additional 4-velocity parameter, so this form of the electrodynamics presumes a relative nature of the mechanical motion; also, this equation describes waves which propagate in space with light velocity kc, which is invariant under the modified Lorentz formulas.
In connection with these two theoretical alternative schemes, an essential issue must be stressed: it seems reasonable to perform the Poincaré-Einstein clock synchronization in uniform media with the help of real light signals influenced by the medium, which leads us to modified Lorentz symmetry. A similar approach is developed for a spin 1/2 particle obeying the Dirac equation in a uniform medium.

Key Words: electromagnetic theory, uniform medium, Minkowski approach, modified Lorentz symmetry.

1 Maxwell equations in a medium, transition to new variables

Maxwell's equations in a uniform medium with two characteristics $\epsilon > 1$ and $\mu > 1$ (dielectric and magnetic penetrabilities) have the form [17]

$$\left\{ \begin{array}{ll} \text{div}\,\mathbf{E} = \dfrac{1}{\epsilon\epsilon_0}\rho \,, & \text{div}\,\mathbf{B} = 0 \,, \\ \dfrac{1}{\mu\mu_0}\text{rot}\,\mathbf{B} = \mathbf{J} + \epsilon\epsilon_0 \dfrac{\partial \mathbf{E}}{\partial t} \,, & \text{rot}\,\mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t} \,. \end{array} \right. \tag{1.1}$$

In the variables

$$\mathbf{B}/\mu\mu_0 = \mathbf{H} \,, \qquad \epsilon\epsilon_0\mathbf{E} = \mathbf{D},$$

the equations (1.1) become

$$\begin{cases} \operatorname{div}\mathbf{D} = \rho\,, & \operatorname{div}\mathbf{H} = 0\,, \\ \operatorname{rot}\mathbf{H} = \mathbf{J} + \dfrac{\partial}{\partial t}\mathbf{D}\,, & \dfrac{1}{\epsilon\epsilon_0}\operatorname{rot}\mathbf{D} = -\mu\mu_0\dfrac{\partial}{\partial t}\mathbf{H}\,. \end{cases} \tag{1.2}$$

The four parameters $\epsilon_0, \mu_0, \epsilon, \mu$ enter the Maxwell equations in the form of two products, $\epsilon_0\epsilon$ and $\mu_0\mu$. This means that besides the charge-current density $(\rho, \mathbf{J})$ and the fields $(\mathbf{E}, \mathbf{B})$ or $(\mathbf{D}, \mathbf{H})$, these equations include only two independent additional parameters. This may be manifestly revealed by introducing two quantities – the light velocity in vacuum c and the (inverse) refraction coefficient k of the medium:

$$c = \frac{1}{\sqrt{\epsilon_0\mu_0}}\,, \qquad k = \frac{1}{\sqrt{\epsilon\mu}} < 1\,.$$

Therefore, from (1.2) we get

$$\begin{cases} \operatorname{div}\mathbf{D} = \rho\,, & \operatorname{div}\mathbf{H} = 0\,, \\ \operatorname{rot}\mathbf{H} = \mathbf{J} + \dfrac{\partial}{\partial t}\mathbf{D}\,, & \operatorname{rot}\mathbf{D} = -\dfrac{1}{k^2c^2}\dfrac{\partial}{\partial t}\mathbf{H}\,. \end{cases} \tag{1.3}$$

The light velocity in a medium c_{med} is less than that in vacuum c and the coefficient k describes this decrease: $c_{med} = kc$. An essential remark is that (1.3) may be re-written as

$$\begin{cases} \operatorname{div}\mathbf{D} = \rho\,, & \operatorname{div}\dfrac{\mathbf{H}}{kc} = 0\,, \\ \operatorname{rot}\dfrac{\mathbf{H}}{kc} = \dfrac{\mathbf{J}}{kc} + \dfrac{\partial}{\partial(kct)}\mathbf{D}\,, & \operatorname{rot}\mathbf{D} = -\dfrac{\partial}{\partial(kct)}\dfrac{\mathbf{H}}{kc}\,. \end{cases} \tag{1.4}$$

Instead of the variables $(t, x^i), (\rho, \mathbf{J}), (\mathbf{D}, \mathbf{H})$, one may define new ones $(x^0, x^i), (\rho, \mathbf{j}), (\mathbf{d}, \mathbf{h})$ by means of the formulas

$$x^0 = kc\,t, \qquad j^0 = \rho, \qquad \mathbf{j} = \frac{\mathbf{J}}{kc}\,, \qquad \mathbf{d} = \mathbf{D}\,, \qquad \mathbf{h} = \frac{\mathbf{H}}{kc}\,.$$

Hence, Maxwell's equations (1.4) will take the form

$$\begin{cases} \operatorname{div}\mathbf{d} = j^0\,, & \operatorname{div}\mathbf{h} = 0\,, \\ \operatorname{rot}\mathbf{h} = \mathbf{j} + \dfrac{\partial}{\partial x^0}\mathbf{d}, & \operatorname{rot}\mathbf{d} = -\dfrac{\partial}{\partial x^0}\mathbf{h}\,. \end{cases} \tag{1.5}$$

The correctness of the relations (1.5) can be additionally checked through dimensional considerations:

$$\begin{cases} [\,x^0\,] = [\,x^i\,] = \text{meter}\,, & [\,\rho\,] = \dfrac{\text{Coulomb}}{\text{meter}^3}\,, \\ [\,j^i\,] = \dfrac{\text{Coulomb}}{\text{meter}^3}\,, & [\,d^i\,] = [\,h^i\,] = \dfrac{\text{Coulomb}}{\text{meter}^2}\,. \end{cases}$$

2 Maxwell's equations, symmetry properties and Lorentz transformations in vacuum and medium

By using the notations $\partial_0 = \dfrac{\partial}{\partial x^0}$, $\partial_i = \dfrac{\partial}{\partial x^i}$, $i = 1, 2, 3$, the Maxwell equations (1.5) can be written down in explicit form as:

$$
\begin{aligned}
(I):\quad & \partial_1\, d^1 + \partial_2\, d^2 + \partial_3\, d^3 = j^{\,0}\ , \\
(II):\quad & \partial_1\, h^1 + \partial_2\, h^2 + \partial_3\, h^3 = 0\ , \\
(III):\quad & \partial_2\, h^3 - \partial_3\, h^2 = j^1 + \partial_0\, d^1\ , \\
& \partial_3\, h^1 - \partial_1\, h^3 = j^2 + \partial_0\, d^2\ , \\
& \partial_1\, h^2 - \partial_2\, h^1 = j^1 + \partial_0\, d^1\ , \\
(IV):\quad & \partial_2\, d^3 - \partial_3\, d^2 = -\,\partial_0\, h^1\ , \\
& \partial_3\, d^1 - \partial_1\, d^3 = -\,\partial_0\, h^2\ , \\
& \partial_1\, d^2 - \partial_2\, d^1 = -\,\partial_0\, h^1\ .
\end{aligned}
\tag{2.1}
$$

Now we introduce a certain linear transformation over the quantities which enter the Maxwell equations (the *modified Lorentz transformation*):

$$
\begin{aligned}
& x^{\,\prime 0} = \mathrm{ch}\sigma\, x^{\,0} - \sinh\sigma\, x^1\ , \quad x'^1 = -\sinh\sigma\, x^{\,0} + \mathrm{ch}\sigma\, x^1\ , \quad x'^2 = x^2\ , \quad x'^3 = x^3\ , \\
& j'^0 = \mathrm{ch}\sigma\, j^{\,0} - \sinh\sigma\, j^1\ , \quad j'^1 = -\sinh\sigma\, j^{\,0} + \mathrm{ch}\sigma\, j^1\ , \quad j'^2 = j^2\ , \quad j'^3 = j^3\ , \\
& d'^1 = +d^1\ , \quad d'^2 = \mathrm{ch}\sigma\, d^2 - \sinh\sigma\, h^3\ , \quad h'^3 = -\sinh\sigma\, d^2 + \mathrm{ch}\sigma\, h^3\ , \\
& h'^1 = +h^1\ , \quad d'^3 = \mathrm{ch}\sigma\, d^3 + \sinh\sigma\, h^2\ , \quad h'^2 = +\sinh\sigma\, d^3 + \mathrm{ch}\sigma\, h^2\ .
\end{aligned}
\tag{2.2}
$$

Now the task is to show that if one transforms (2.1) to the new (primed) variables

$$
(x^{\,0}, x^{\,i}),\ (j^{\,0}, j^{\,i}),\ (d^{\,i},\ h^{\,i}) \quad \Longrightarrow \quad (x'^0, x^{\,\prime i}),\ (j'^0, j^{\,\prime i}),\ (d^{\,\prime i},\ h^{\,\prime i})\ ,
$$

then as a result one will obtain again equations of the form (2.1), with a single difference: all quantities become primed ones. In essence, this is the main statement of Lorentz and Poincaré on the symmetry properties of Maxwell theory [5, 14, 15]. Before proceeding further, we derive transforms for the derivatives with respect to the coordinates:

$$
\partial_0 = \cosh\sigma\, \partial'_0 - \sinh\sigma\, \partial'_1\ , \quad \partial_1 = -\sinh\sigma\, \partial'_0 + \cosh\sigma\, \partial'_1\ , \quad \partial_2 = \partial_{\prime 2}\ , \qquad \partial_3 = \partial_{\prime 3}\ ,
$$

We further apply specific mathematical techniques: after simple calculations we infer

$$
\begin{cases}
(I) & \cosh\sigma[(\partial_{\prime 1} d^{\prime 1} + \partial_{\prime 2} d^{\prime 2} + \partial_{\prime 3} d^{\prime 3}) - j^{\prime 0}] - \sinh\sigma[-(\partial_{\prime 2} h^{\prime 3} - \partial_{\prime 3} h^{\prime 2}) + j^{\prime 1} + \partial_{\prime 0} d^{\prime 1}] = 0, \\
(II) & \cosh\sigma(\partial_{\prime 1} h^{\prime 1} + \partial_{\prime 2} h^{\prime 2} + \partial_{\prime 3} h^{\prime 3}) - \sinh\sigma[\partial'_0 h^{\prime 1} + (\partial_{\prime 2} d^{\prime 3} - \partial_{\prime 3} d^{\prime 2})] = 0, \\
(III)_1 & \cosh\sigma[(\partial_{\prime 2} h^{\prime 3} - \partial_{\prime 3} h^{\prime 2}) - j^{\prime 1} - \partial_{\prime 0} d^{\prime 1}] - \sinh\sigma[-(\partial_{\prime 1} d^{\prime 1} + \partial'_2 d^{\prime 2} + \partial_{\prime 3} d^{\prime 3}) + j^{\prime 0}] = 0, \\
(III)_2 & \partial_{\prime 3} h^{\prime 1} - \partial_{\prime 1} h^{\prime 3} = j^{\prime 2} + \partial_{\prime 0} d^{\prime 2}\ , \\
(III)_3 & \partial_{\prime 1} h^{\prime 2} - \partial_{\prime 2} h^{\prime 1} = j^{\prime 3} + \partial_{\prime 0} d^{\prime 3}, \\
(IV)_1 & \cosh\sigma[(\partial_{\prime 2} d^{\prime 3} - \partial_{\prime 3} d^{\prime 2}) + \partial_{\prime 0} h^{\prime 1}] - \sinh\sigma(\partial_{\prime 1} h^{\prime 1} + \partial_{\prime 2} h^{\prime 2} + \partial_{\prime 3} h^{\prime 3}) = 0, \\
(IV)_2 & \partial_{\prime 3} d^{\prime 1} - \partial_{\prime 1} d^{\prime 3} = -\partial_{\prime 0} h^{\prime 2}, \\
(IV)_3 & \partial_{\prime 1} d^{;2} - \partial_{\prime 2} d^{\prime 1} = -\partial_{\prime 0} h^{\prime 3}.
\end{cases}
$$

Let us combine (II) and $(IV)_1$:

$$\cosh\,\sigma\,(II) + \sinh\,\sigma\,(IV)_1 = (II)' \quad \Longrightarrow \partial_{\,'1}\, h^{\,'1} + \partial_{\,'2}\, h^{\,'2} + \partial_{\,'3}\, h^{\,'3} = 0\;,$$
$$\sinh\,\sigma\,(II) + \cosh\,\sigma\,(IV)_1 = (IV)'_1 \quad \Longrightarrow \partial'_2\, d'^3 - \partial'_3\, d'^2 = -\,\partial'_0\, h'^1\;.$$

Analogously, by combining (I) and $(III)_1$, we get:

$$\cosh\,\sigma\,(I) + \sinh\,\sigma\,(III)_1 = (I)' \quad \Longrightarrow \partial'_1\, d'^1 + \partial'_2\, d'^2 + \partial'_3\, d'^3 = j'^0\;,$$
$$\sinh\,\sigma\,(I) + \cosh\,\sigma\,(III)_1 = (IV)'_1 \quad \Longrightarrow \partial'_2\, d'^3 - \partial'_3\, d'^2 = -\,\partial'_0\, h'^1\;.$$

Conclusion. From the previous there follow the Maxwell equations in primed variables

$$(I):\quad \partial'_1\, d'^1 + \partial'_2\, d'^2 + \partial'_3\, d'^3 = j'^0\;,$$

$$(II):\quad \partial_{\,'1}\, h^{\,'1} + \partial_{\,'2}\, h^{\,'2} + \partial_{\,'3}\, h^{\,'3} = 0\;,$$

$$\begin{aligned}(III):\quad &\partial_{\,'2}\, h^{\,'3} - \partial_{\,'3}\, h^{\,'2} = j^{\,'1} + \partial_{\,'0}\, d^{\,'1}\;,\\ &\partial_{\,'3}\, h^{\,'1} - \partial_{\,'1}\, h'^3 = j'^2 + \partial'_0\, d'^2\;,\\ &\partial'_1\, h'^2 - \partial'_2\, h'^1 = j'^1 + \partial'_0\, d'^1\;,\end{aligned}$$

$$\begin{aligned}(IV):\quad &\partial'_2\, d'^3 - \partial'_3\, d'^2 = -\,\partial'_0\, h'^1\;,\\ &\partial'_3\, d^{\,'1} - \partial'_1\, d'^3 = -\,\partial'_0\, h'^2\;,\\ &\partial'_1\, d'^2 - \partial'_2\, d'^1 = -\,\partial'_0\, h'^1\;,\end{aligned}$$

or, in vector form,

$$\begin{cases} \text{div}'\,\mathbf{d}' = j^{\,'0}\;, & \text{div}'\,\mathbf{h}' = 0\;,\\ \text{rot}'\,\mathbf{h}' = \mathbf{j}' + \dfrac{\partial}{\partial x'^0}\,\mathbf{d}'\;, & \text{rot}'\,\mathbf{d}' = -\,\dfrac{\partial}{\partial x'^0}\,\mathbf{h}'\;. \end{cases} \tag{2.3}$$

3 On the physical interpretation of Lorentz transformation for Maxwell equations in vacuum and medium

In understanding the physical sense of the dimensionless parameter σ in the above Lorentz formulas, one notices that a more in-depth question might be: how do the Maxwell equations behave when the reference frame changes from K to a moving one, K'. For the situation when velocity is small enough, we must obtain a simple and «evident» solution in the form of Galileo formula for a coordinate transform:

$$t' = t\;,\qquad x' = x - Vt\;,\qquad y' = y\;,\qquad z' = z\;.$$

Let us consider the Lorentz transformation

$$x'^0 = \cosh\sigma\, x^{\,0} - \sinh\sigma\, x^1\;,\qquad x'^1 = -\sinh\sigma\, x^{\,0} + \cosh\sigma\, x^1\;,\qquad x'^2 = x^2\;,\; x'^3 = x^3$$

at a very small σ. We consider the Taylor expansions

$$e^{+\sigma} = 1 + \frac{\sigma}{1!} + \frac{\sigma^2}{2!} + ...,\; e^{-\sigma} = 1 - \frac{\sigma}{1!} + \frac{\sigma^2}{2!} - ...,$$

and at $\sigma \ll 1$ we get the approximating formulas

$$\cosh\,\sigma = 1 + \frac{\sigma^2}{2!} + ... \approx 1,\qquad \sinh\,\sigma = \frac{\sigma}{1!} + \frac{\sigma^3}{3!} + ... \approx \sigma.$$

Thus, *the Lorentz transformation at small* σ will take the form

$$kct' \approx kct - \sigma x \approx kct,$$

whence $t' = t$ and $x' = -\sigma kct + x = x - Vt$, if $\sigma = \dfrac{V}{kc}$. Hence the physical sense of the parameter σ (at its small values $\sigma \ll 1$) is expressed by:

$$\sigma \ll 1 \qquad \Longrightarrow \qquad \sigma = \frac{V}{kc} = \frac{V}{c_{med}} \ . \tag{3.1}$$

One might further need to generalize (3.1) for arbitrary values of V. This is achieved by the following relations:

$$\begin{aligned} &0 < \mid V \mid < kc \qquad \Longleftrightarrow \qquad 0 < \mid \sigma \mid < 1 \ , \\ &\cosh\, \sigma = \frac{1}{\sqrt{1-(V/kc)^2}} \ , \qquad \sinh\, \sigma = \frac{(V/kc)}{\sqrt{1-(V/kc)^2}} \ . \end{aligned} \tag{3.2}$$

Evidently, at small V, (3.2) coincide with (3.1). It is useful to *translate the Lorentz transformations to ordinary units*:

$$t' = \frac{t - Vx/k^2c^2}{\sqrt{1-(V/kc)^2}} \ , \qquad x' = x - \frac{Vt}{\sqrt{1-(V/kc)^2}} \ .$$

4 Behavior of the light velocity under modified Lorentz transformations

While finding the modified Lorentz transformations, *a simple kinematical problem of the modified Lorentz formulas for velocity* may be immediately solved. This provides us with a postulate on the constancy of light velocity (kc), the crucial logical element in Einstein's construction of Special Relativity.

Let us observe a material point in a fixed (non-moving) reference frame K, which starts its history at

$$t_1 = 0 \ , \ \ x_1 = 0 \ , \ \ y_1 = 0 \ , \ \ z_1 = 0$$

and moves in the plane (x, y) along direction $(\cos\phi, \sin\phi)$ with velocity

$$W_x = W\cos\phi \ , \qquad W_y = W\sin\phi \ , \qquad W_z = 0 \ , \qquad W = \sqrt{W_x^2 + W_y^2} \ .$$

At the moment $t_2 > 0$, its coordinates become

$$t_2, \quad x_2 = W\cos\phi\, t_2 \ , \quad y_2 = W\sin\phi\, t_2, \quad z_2 = 0 \ .$$

The same can be re-written in the form

$$W_x = \frac{x_2 - x_1}{t_2 - t_1} \ , \qquad W_y = \frac{y_2 - y_1}{t_2 - t_1} \ , \qquad W_z = 0 \ .$$

Let us find how these velocity components behave under the modified Lorentz transformation. To this end, we should calculate

$$W'_x = \frac{x'_2 - x'_1}{t'_2 - t'_1} \ , \qquad W'_y = \frac{y'_2 - y'_1}{t'_2 - t'_1} \ ,$$

With the use of

$$t'_1 = 0 \,, \qquad x'_1 = 0 \,, \qquad t'_2 = \frac{t_2 - V x_2/k^2c^{\,2}}{\sqrt{1 - V^2/k^2c^{\,2}}} \,, \qquad x'_2 = \frac{x_2 - V t_2}{\sqrt{1 - V^2/k^2c^{\,2}}} \,,$$

for W'_x we get

$$W'_x = \frac{x'_2 - x'_1}{t'_2 - t'_1} = \frac{(\, x_2 - V t_2 \,)}{(\, t_2 - V x_2/k^2c^{\,2} \,)} = \frac{(\, x_2 - x_1) - V(t_2 - t_1 \,)}{(t_2 - t_1) - V(x_2 - x_1/k^2c^{\,2} \,)} = \frac{W_x - V}{1 - V W_x/k^2c^2} \,.$$

Now we turn to W'_y:

$$W'_y = \frac{y'_2 - y'_1}{t'_2 - t'_1} = \frac{y_2}{t'_2} = \sqrt{1 - V^2/k^2c^{\,2}} \; \frac{W_y t_2}{t_2 - V \; x_2/k^2c^{\,2}} = \frac{\sqrt{1 - V^2/k^2c^{\,2}} \cdot W_y(t_2 - t_1)}{(t_2 - t_1) - V \; (x_2 - x_1)/k^2c^{\,2}},$$

whence,

$$W'_y = \frac{\sqrt{1 - V^2/k^2c^{\,2}}}{1 - \; V W_x/k^2c^2} \, W_y \,.$$

For the W_z component, we have a trivial result:

$$W'_z = \frac{z'_2 - z'_1}{t'_2 - t'_1} = \frac{z_2 - z_1}{t'_2 - t'_1} = 0 = W_z \,.$$

Thus, the velocity vector $\mathbf{W} = (W_x, W_y, 0)$ transforms as follows

$$W'_x = \frac{W_x - V}{1 - V W_x/k^2c^2} \,, \qquad W'_y = \frac{\sqrt{1 - V^2/k^2c^{\,2}}}{1 - V W_x/k^2c^2} \, W_y \,, \qquad W'_z = W_z = 0 \,. \tag{4.1}$$

This is *a modified version of the famous rule for velocity summing by Lorentz-Poincaré-Einstein.*

For simplifying the appearance of many formulas to come, we will make formal changes [1]

$$\frac{V}{kc} \rightsquigarrow V, \qquad \frac{W_x}{kc} \rightsquigarrow W_x, \quad \text{etc},$$

Then (4.1) become

$$W'_x = \frac{W_x - V}{1 - V W_x} \,, \qquad W'_y = \frac{\sqrt{1 - V^2}}{1 - V W_x} \; W_y \,, \qquad W'_z = W_z \,. \tag{4.2}$$

We shall further point out a series of *consequences* which follow from the developed framework.

By applying (4.2) to the light ray propagating along the x axis in fixed (unmoving) reference frame K, (4.2) provide us with simple but striking result:

Consequence 1. *The light ray propagating along the x axis in the moving reference frame K' has the velocity*

$$\left\{ \begin{array}{l} W_x = 1 \\ W_y = 0 \\ W_z = 0 \end{array} \right. \qquad \Longrightarrow \qquad \left\{ \begin{array}{l} W'_x = \dfrac{1}{1 - V W_x} \, (\, W_x - V \,) = \dfrac{1 - V}{1 - V} = 1 \,, \\ W'_y = \dfrac{\sqrt{1 - V^2}}{1 - V W_x} \; W_y \; = 0, \\ W'_z = 0 \,, \end{array} \right.$$

that is

$$W'_x = 1 \,, \qquad W'_y = 0 \,, \qquad W'_z = 0 \,.$$

[1] Sometimes this will be formulated as the use of a special system with unit light velocity.

The Lorentz transformation along the x axis does not change the speed of light along the direction of x.

By applying the (4.2) to a light ray propagating along arbitrary ϕ-direction in a fixed (non-moving) reference frame K, and using Eqs. (4.2), we infer

Consequence 2. *The velocity vector in the moving reference frame K' is given by*

$$\begin{cases} W_x^2 + W_y^2 = 1, \\ W_z = 0 \end{cases} \Rightarrow \begin{cases} W'_x = \dfrac{1}{1 - VW_x}\,(W_x - V), \\ W'_y = \dfrac{\sqrt{1-V^2}}{1 - VW_x}\; W_y\,, \\ W'_z = 0\,, \end{cases}$$

and for the squared norm of the vector velocity, we get

$$W'^{\,2}_x + W'^{\,2}_y = \frac{W_x^2 - 2VW_x + V^2 + (1-V^2)(1-W_x^2)}{(1-VW_x)^2} = +1.$$

The Lorentz transformation along the axis x does not change the norm of the light velocity vector.

Consequence 3. *We note that the same result holds true even for $W_z \neq 0$, since*

$$(W_x^2 + W_y^2) + W_z^2 = 1 \;\;\Rightarrow\;\; W'^{\,2}_x + W'^{\,2}_y + W'^{\,2}_z = \frac{W_x^2 - 2VW_x + V^2 + (1-V^2)(1-W_x^2)}{(1-VW_x)^2} = 1\,.$$

Consequence 4. *The invariance of the norm of the velocity vector under Lorentz transformations concerns only the light velocity-which has unit length in the fixed (non-moving) frame.* We have, indeed,

$$W_x^2 + W_y^2 = W^2, \;\; W_z = 0\,,$$

whence

$$W'^2 = \frac{W_x^2 - 2VW_x + V^2 + (1-V^2)(W^2 - W_x^2)}{(1-VW_x)^2} = 1 + \;(W^2 - 1)\frac{1-V^2}{(1-VW_x)^2}\,.$$

Consequence 5. The light aberration. *The above formulas of the velocity vector transforms can be expressed in terms of angular variables. In the case of the light,*

$$(W_x, W_y, 0)\,, \qquad W_x^2 + W_y^2 = 1\,, \qquad \cos\phi = W_x \qquad \sin\phi = W_y\,,$$

so that from (4.2) there follow the formulas for the light aberration,

$$\cos\phi' = \frac{\cos\phi - V}{1 - V\cos\phi}\,, \qquad \sin\phi' = \sin\phi\frac{\sqrt{1-V^2}}{1 - V\cos\phi}\,.$$

5 The Lorentz-Poincaré-Einstein controversy

It was Lorentz [5] who first established a remarkable property of Maxwell equations, namely its (approximate) symmetry under special mathematical transformations when plugging into these equations quantities as time and space coordinates, charge-current density, and electromagnetic fields.

Poincaré introduced exactness and clarity [14, 15] into the Lorentz initial formulas, and revealed its mathematical (so-called) group structure. Undoubtedly, the first deciding steps on the road to Special Relativity theory were made by Lorentz and this was stressed by Poincaré more than once. At the same time, Lorentz never ascribes to himself the merit all along this road, and willingly appreciated the role of Poincaré's contribution.

Unfortunately, afterwards, in connection with Special Relativity, there arose both controversy and misunderstanding on the question – who is the creator of this theory: Lorentz, Poincaré, or Einstein [2, 3]. To the present day this dispute is still present. We shall not join any side of the debaters, and assert that in our opinion all three, Lorentz, Poincaré, and Einstein, are the creators of the theory[2].

The first was Lorentz, then Poincaré sided with him, and next Einstein started his work on creating Special Relativity [2, 3], mainly on its physical interpretation, comprehension, and logical reconstruction. The question – who is the main *creator – is false. Lorentz formulated his view on this matter concisely and definitely [18–22]* the same what we had deduced from Maxwell's equations, Einstein has postulated ....

So, the development of Special Relativity arose along two lines, which are absurd to be consider as absolutely independent. One line goes upward to Special Relativity from the symmetry property of the Maxwell Theory in moving bodies. This is an inductive *way and it is historically the first line.*

The second line, though logically independent in appearance, is a deductive *construction of the theory by going down from a special postulate [2, 3]. But the postulate itself can be regarded as a logical mathematical result of the Lorentz-Poincaré analysis of Maxwell's theory. The logical treatment suggested by Einstein seems for many people simple and clear, so that it may be explained even to a person without any special education. This circumstance assists in the promotion of Einstein's treatment of Special Relativity and its notability for the general public.*

However, the creating by deductive way to construct Special Relativity does not provide grounds to assign to A. Einstein the main or single creator role for the theory: there were Lorentz and Poincaré who provided us as well with the inductive way to this theory — and this way was historically the first. Both approaches to Special Relativity are legitimate and mutually complementary.

6 Deriving the Lorentz formulas from Einstein postulate of (modified) light velocity constancy

Consider the known Einstein postulate:

The (modified) light velocity is the same in both reference frames, K (the fixed frame) and the other one K' (which moves with velocity V).

Problem.

What kind of linear transformation between (t, x) and (t', x') agrees with the postulate of (modified) light velocity constancy.

The invariance condition for the light velocity is

$$kc = \frac{x_2 - x_1}{t_2 - t_1}\,, \qquad kc = \frac{x'_2 - x'_1}{t'_2 - t'_1}\,,$$
$$0 = k^2c^2(t_2 - t_1)^2) - (x_2 - x_1)^2 = k^2c^2(t'_2 - t'_1)^2) - (x'_2 - x'_1)^2\,.$$

[2]We have to give credit to a great number of physicists as well and their enormous and laborious work, still emphasizing that these three great names are the historic epistemic promoters.

We enforce the following special convention: let the light ray start propagating at $t_1 = 0, x_1 = 0$ in the positive direction of the axis x in the K-frame. Due to the linearity of the transformation law, we have in the reference frame k' that $t'_1 = 0$, $x'_1 = 0$, as well. Therefore, the previous relation will take the form[3]:

$$k^2c^2t^2 - x^2 = k^2c^2(t')^2 - (x')^2 \ , \qquad (x^0)^2 - x^2 = (x'^0)^2 - (x')^2 \ .$$

So, the task is to find a linear transformation which leaves invariant such an unusual length of the vector (x^0, x). However, the solution to this problem is already known, and was given by Lorentz:

$$\begin{cases} x'^0 = \cosh\sigma\, x^{\,0} - \sinh\sigma\, x \ , \\ x^{\,\prime} = -\sinh\sigma\, x^{\,0} + \cosh\sigma\, x \ , \end{cases}$$

or, using (3.2)

$$t' = \frac{t - Vx/k^2c^2}{\sqrt{1 - V^2/k^2c^2}} \ , \qquad x' = \frac{x - Vt}{\sqrt{1 - V^2/k^2c^2}}.$$

In other words, there exists a logical relationship between the Einstein postulate of (modified) light velocity constancy and the (modified) Lorentz formulas for coordinate transformations. Here it is appropriate to remember the words of Lorentz [18–22]: *the same what we had deduced from Maxwell's equations Einstein has postulated.* Evidently, this is exactly the case.

7 On Maxwell theory in a uniform medium - the Minkowski approach

Although A. Einstein's main work on Special Relativity [2, 3] dated 1905 is titled "*On the Electrodynamics of Moving Bodies*", in this paper Maxwell equations had been considered only in vacuum[4] in fact, and the symmetry properties of these equations have been used. In accordance with this, throughout the whole theoretical construction, from the very beginning, only a certain universal light velocity in vacuum was used and only for that very velocity was advanced a postulate regarding its constancy irrespective of the motion of the reference frame.

Later in 1908, H. Minkowski gave [7–11] a more detailed and accurate treatment of the Maxwell Theory in a uniform medium ($\epsilon \neq 1, \mu \neq 1$) with respect to the requirements of Special Relativity. Two issues of his study should be emphasized:

Minkowski elaborated a very convenient and still actively exploited mathematical technique – the so called 4-dimensional tensor formalism[5]. *Minkowski found the way to describe symmetry properties of the Maxwell equations in a uniform medium with the use of Lorentz formulas on the base of the light velocity c in vacuum*[6]. *In this work Minkowski achieved in fact a certain unification between Einstein's earlier analysis and electrodynamics in medium.*

Here it might be specially mentioned that the logical construction of Special Relativity by Einstein does not formally depend on the numerical value of light velocity – this might be 300.000 km/sec as well as 3 sm/sec. The essential thing here is only the existence of a (light) signal which goes through the space with the same velocity, for all inertial observers. Moreover, in this context we should admit that operating with a light signal of velocity c in a medium is a fiction; in fact any real light can move through a uniform medium with velocity $\tilde{c} = kc$. So it might seem as well-taken the requirement to perform clock synchronization in uniform media

[3]Here, the index 2 is omitted in x'_2 and in x_2.

[4]A medium with trivial values $\epsilon = 1, \mu = 1$.

[5]To be exact, H.Poincaré had proposed and developed in some aspects the same technique before Minkowski [[5].

[6]This issue is most significant in the context established above: the presence of symmetry of the Maxwell theory in a medium under modified Lorentz transformations involving the light velocity kc in the medium.

with the help of real light signals influenced by the medium. However this was not done neither by Einstein, nor by Minkowski. On the contrary, Minkowski found the way to speak about relativistic symmetry of Maxwell Theory in a medium and to use only the Lorentz formulas with the vacuum light velocity c.

Below we shall introduce *Minkowski's approach* [7–11] without following it in detail.

8 Standard Lorentz symmetry of the Maxwell equations in a medium

We start from *the Maxwell equations* in the form

$$\begin{cases} \text{div } \mathbf{D} = J^0 \,, \quad \text{rot } \dfrac{\mathbf{H}}{c} = \dfrac{\mathbf{J}}{c} + \dfrac{\partial \mathbf{D}}{\partial ct} \,, \\ \text{div } c\mathbf{B} = 0 \,, \quad \text{rot } \mathbf{E} = - \dfrac{\partial c\mathbf{B}}{\partial ct} \,, \end{cases} \tag{8.1}$$

with $(x^0, x^i) = (ct, x^i)$, $(J^0 = \rho, \dfrac{\mathbf{J}}{c})$, and where

- $\mathbf{D}$ is the electric displacement field,
- $\mathbf{H}$ is the magnetizing field,
- $\mathbf{E}$ is the electric field,
- $\mathbf{B}$ is the magnetic field,
- $\mathbf{J}$ is the total current density,
- ρ is the total charge density,
- t is time,
- c is the speed of light in vacuum.

Here the equations are divided into two groups, regarding the vectors $(\mathbf{D}, \mathbf{H}/c)$ and the vectors $(\mathbf{E}, c\mathbf{B})$. Note that the source fields $(J^0 = \rho, \mathbf{J}/c)$ enter only the first group. Also, one issue to emphasize is that (8.1) do not include the parameters of permittivity and of permeability of the free space.

dielectric and magnetic penetrability; however as a peculiar compensation for this, we need to simultaneously use two sets of electromagnetic vectors: $(\mathbf{D}, \mathbf{H}/c)$ and $(\mathbf{E}, c\mathbf{B})$.

It is readily established that if (8.1) are subjected to the (ordinary) Lorentz transformation (with the light velocity c in the vacuum and correspondingly with the variable $x^0 = ct$)

$$\begin{aligned} & x'^0 = \cosh\beta\, x^0 - \sinh\beta\, x^1 \,, \quad x'^1 = -\sinh\beta\, x^0 + \cosh\beta\, x^1 \,, \quad x'^2 = x^2 \,,\ x'^3 = x^3 \,, \\ & J'^0 = \cosh\beta\, J^0 - \sinh\beta\, c^{-1} J^1 \,, \quad c^{-1} J'^1 = -\sinh\beta\, J^0 + \cosh\beta\, c^{-1} J^1 \,, \quad J'^2 = J^2 \,,\ J'^3 = J^3 \,, \\[2ex] & D'^1 = +D^1 \,, \quad D'^2 = \cosh\beta\, D^2 - \sinh\beta\, c^{-1} H^3 \,, \quad D'^3 = \cosh\beta\, D^3 + \sinh\beta\, c^{-1} H^2 \,, \\ & H'^1 = +H^1 \,, \quad c^{-1} H'^2 = +\sinh\beta\, D^3 + \cosh\beta\, c^{-1} H^2 \,, \quad c^{-1} H'^3 = -\sinh\beta\, D^2 + \cosh\beta\, c^{-1} H^3 \,, \\ & E'^1 = +E^1 \,, \quad E'^2 = \cosh\beta\, E^2 - \sinh\beta\, cB^3 \,, \quad E'^3 = \cosh\beta\, E^3 + \sinh\beta\, cB^2 \,, \\ & B'^1 = +B^1 \,, \quad cB'^2 = +\sinh\beta\, E^3 + \cosh\beta\, cB^2 \,, \quad cB'^3 = -\sinh\beta\, E^2 + \cosh\beta\, cB^3 \,, \end{aligned} \tag{8.2}$$

where

$$\cosh\, \beta = \frac{1}{\sqrt{1-(V/c)^2}} \,, \qquad \sinh\, \beta = \frac{(V/c)}{\sqrt{1-(V/c)^2}} \,,$$

we shall again obtain equations in Maxwell's form:

$$\begin{cases} \text{div}'\,\mathbf{D}' = J'^{0}\,, \quad \text{rot}'\,\dfrac{\mathbf{H}'}{c} = \dfrac{\mathbf{J}'}{c} + \dfrac{\partial \mathbf{D}'}{\partial ct'}\,, \\ \text{div}'\,c\mathbf{B}' = 0\,, \quad \text{rot}'\,\mathbf{E}' = -\,\dfrac{\partial c\mathbf{B}'}{\partial ct'}\,. \end{cases}$$

The issue of first importance is that the modified Lorentz transformations used in (2.2) generate significantly different formulas. For convenience, we shall write them down[7]; from formal viewpoint, the whole difference reduces to the emerging modified quantity $\tilde{c} = kc$):

The modified relations (the light velocity *in a medium* kc *and* $x^0 = kct$).

$$x'^0 = \cosh\sigma\; x^{\,0} - \sinh\sigma\; x^1\,,\; x'^1 = -\sinh\sigma\; x^{\,0} + \cosh\sigma\; x^1\,,\; x'^2 = x^2\,,\; x'^3 = x^3\,,$$

$$J'^0 = \cosh\sigma\; J^{\,0} - \sinh\sigma\;(kc)^{-1}J^1\,,\; (kc)^{-1}J'^1 = -\sinh\sigma\; J^{\,0} + \cosh\sigma\;(kc)^{-1}J^1\,,\; J'^2 = J^2\,,\; J'^3 = J^3\,,$$
$$D'^1 = +D^1\,,\; D'^2 = \cosh\sigma\; D^2 - \sinh\sigma\;(kc)^{-1}H^3\,,\; D'^3 = \cosh\sigma\; D^3 + \sinh\sigma\;(kc)^{-1}H^2\,,$$
$$H'^1 = +H^1\,,\; (kc)^{-1}H'^2 = +\sinh\sigma\; D^3 + \cosh\sigma\;(kc)^{-1}H^2\,,\; (kc)^{-1}H'^3 = -\sinh\sigma\; D^2 + \cosh\sigma\;(kc)^{-1}H^3\,,$$
$$E'^1 = +E^1\,,\; E'^2 = \cosh\sigma\; E^2 - \sinh\sigma\; kcB^3\,,\; E'^3 = \cosh\sigma\; E^3 + \sinh\sigma\; kcB^2\,,$$
$$B'^1 = +B^1\,,\; kcB'^2 = +\sinh\sigma\; E^3 + \cosh\sigma\; kcB^2\,, kcB'^3 = -\sinh\sigma\; E^2 + \cosh\sigma\; kcB^3\,,$$

where $\cosh\,\sigma = \dfrac{1}{\sqrt{1-(V/kc)^2}}$, $\sinh\,\beta = \dfrac{(V/kc)}{\sqrt{1-(V/kc)^2}}$.

So, regarding the Maxwell equations we face the rather peculiar situation, in which two different symmetries are revealed at the same time:

- a symmetry with respect to the ordinary Lorentz transformations L, in which there appears a universal constant – the light velocity in the vacuum c;
- another symmetry, with respect to the modified Lorentz transformations L^{mod}, in which there appears a medium dependent constant – the light velocity kc in the medium.

The explicit transforms both for space-time coordinates and for electromagnetic quantities

$$(t, x^i)\,, \qquad (J^0, J^i)\,, \qquad (E^i, B^i)\,, \qquad (D^i, H^i)$$

differ for these two cases. Then there occur the following open questions:

- Which symmetry of these two is more adequate to consider?
- Which is the in-depth meaning of the simultaneous existence of two symmetries for Maxwell equations in a medium?
- Which of them closer corresponds to the physical reality?
- Do there exist any criteria to pick out only one of two logical possibilities?

From purely theoretical view point, considering the need to synchronize clocks with the help of real light signals in a medium, as being imperative one must use the modified version of the Lorentz transformations in the medium.

[7] In order to to distinguish them, instead of β we shall use the symbol σ.

9 The constitutive conditions $\mathbf{D} = \epsilon_0\epsilon\mathbf{E}$, $\mathbf{B} = \mu_0\mu\mathbf{H}$, and the Minkowski equations

We further examine the following problem: which will be the form off the field relations

$$D^i = \epsilon_0\epsilon\ E^i, \qquad H^i = \frac{1}{\mu_0\mu}\ B^i\ , \tag{9.1}$$

take after the Lorentz transformation to a moving reference frame? This problem was firstly considered by H. Minkowski in 1908 [7–11]. For simplicity we will consider the simplest Lorentz formulas that correspond to a moving reference frame along the x axis.

We first consider the ordinary Lorentz transforms. With the use of (8.2), from (9.1) it follows

$$D^i = \epsilon_0\epsilon\ E^i \implies \begin{cases} D'^1 = \epsilon_0\epsilon\ E'^1\ , \\ \cosh\beta\ D'^2 + \sinh\beta\ \dfrac{H'^3}{c} = \epsilon_0\epsilon\ (\cosh\beta\ E'^2 + \sinh\beta\ cB'^3)\ , \\ \cosh\beta\ D'^3 - \sinh\beta\ \dfrac{H'^2}{c} = \epsilon_0\epsilon\ (\cosh\beta\ E^{;'3} - \sinh\beta\ cB'^2); \end{cases} \tag{9.2}$$

$$H^i = \frac{1}{\mu_0\mu}\ B^i \implies \begin{cases} H'^1 = \dfrac{1}{\mu_0\mu}\ B^1\ , \\ \sinh\beta\ D'^3 - \cosh\beta\ \dfrac{H'^2}{c} = \dfrac{1}{\mu_0\mu}\dfrac{1}{c^2}\ (\sinh\beta\ E'^3 - \cosh\beta\ cB'^2)\ , \\ \sinh\beta\ D'^2 + \cosh\beta\ \dfrac{H'^3}{c} = \dfrac{1}{\mu_0\mu}\dfrac{1}{c^2}\ (\sinh\beta\ E'^2 + \cosh\beta\ cB'^3); \end{cases} \tag{9.3}$$

we notice that

$$\frac{1}{\mu_0\mu c^2} = \frac{\epsilon_0\mu_0}{\mu_0\mu} = \frac{\epsilon_0\epsilon}{\epsilon\mu} = \epsilon_0\epsilon\ k^2\ .$$

The relations (9.2) and (9.3) are just what we call *Minkowski equations* [7–11] written down in a particular simple case. Let us change them to another form. To this end, they should be rewritten as three pairs of linear systems in the variables (D'^1, H'^1), $(D'^2, H'^3/c)$, $(D'^3, H'^2/c)$:

$$\begin{aligned} & D'^1 = \epsilon_0\epsilon\ E'^1\ , \qquad H'^1 = \frac{1}{\mu_0\mu}\ B^1\ ; \\ & \cosh\beta\ D'^2 + \sinh\beta\ H'^3/c = \epsilon_0\epsilon(\cosh\beta\ E'^2 + \sinh\beta\ cB'^3)\ , \\ & \sinh\beta\ D'^2 + \cosh\beta\ H'^3/c = \epsilon_0\epsilon k^2(\sinh\beta\ E'^2 + \cosh\beta\ cB'^3)\ ; \\ & \cosh\beta\ D'^3 - \sinh\beta\ \frac{H'^2}{c} = \epsilon_0\epsilon(\cosh\beta\ E'^3 - \sinh\beta\ cB'^2)\ , \\ & \sinh\beta\ D'^3 - \cosh\ \beta H'^2/c = \epsilon_0\epsilon k^2(\sinh\beta\ E'^3 - \cosh\beta\ cB'^2)\ . \end{aligned} \tag{9.4}$$

The solutions of (9.4)[8] are

$$\begin{aligned} & \begin{cases} D'^1 = \epsilon_0\epsilon\ E'^1\ , \\ D'^2 = \epsilon_0\epsilon\ [\ (\cosh^2\beta - k^2\sinh^2\beta)\ E'^2 + \sinh\beta\ \cosh\beta\ (1-k^2)\ cB'^3]\ , \\ D'^3 = \epsilon_0\epsilon\ [\ (\cosh^2\beta - k^2\sinh^2\beta)\ E'^3 - \sinh\beta\ \cosh\beta\ (1-k^2)\ cB'^2\ ]\ , \end{cases} \\ & \begin{cases} H'^1/c = \epsilon_0\epsilon k^2\ cB^1\ , \\ H'^2/c = \epsilon_0\epsilon\ [\ (k^2\cosh^2\beta - \sinh^2\beta)\ cB'^2 - \sinh\beta\ \cosh\beta\ (k^2-1)\ E'^3\ ]\ , \\ H'^3/c = \epsilon_0\epsilon\ [\ (k^2\ \cosh^2\beta - \sinh^2\beta)\ cB'^3 + \sinh\beta\ \cosh\beta\ (k^2-1)\ E'^2\ ]\ . \end{cases} \end{aligned} \tag{9.5}$$

[8] These are the same Minkowski relations only translated to another form.

The relations (9.5) say that the simple connections (9.1) which exist between electromagnetic vectors in initial (non-moving) frame after translating to a moving reference frame become rather complex ones: they involve now the velocity as a parameter. In other terms, this means that the field relations (9.1) are not Lorentz invariant.

However, we can see that *in the vacuum case when* $k = 1$, the formulas (9.5) will take the same simplest form from which we initially started:

$$k = 1 \; , \;\; D'^{i} = \epsilon_0 \, E'^{i} \; , \;\; H'^{i} = \frac{1}{\mu_0} \, B^{i} \; . \tag{9.6}$$

Now, we consider the field relations, while using *modified Lorentz transformations.* We shall easily see that *the properties of these relations under the modified Lorentz theory are different and much more attractive: they turn out to be Lorentz invariant.*

Indeed, let us start with[9]:

$$D^i = \epsilon_0 \epsilon \, E^i \quad \Longrightarrow \quad \begin{cases} D'^1 = \epsilon_0 \epsilon \, E'^1 \; , \\ \cosh\sigma \, D'^2 + \sinh\sigma \, \dfrac{H'^3}{kc} = \epsilon_0 \epsilon \, (\cosh\sigma \, E'^2 + \sinh\sigma \, kcB'^3) \; , \\ \cosh\sigma \, D'^3 - \sinh\sigma \, \dfrac{H'^2}{kc} = \epsilon_0 \epsilon \, (\cosh\sigma \, E'^3 - \sinh\sigma \, kcB'^2) \; ; \end{cases}$$

$$H^i = \frac{1}{\mu_0 \mu} \, B^i \quad \Longrightarrow \quad \begin{cases} H'^1 = \dfrac{1}{\mu_0 \mu} \, B'^1 \; , \\ -\sinh\sigma \, D'^3 + \cosh\sigma \, \dfrac{H'^2}{kc} = \dfrac{1}{\mu_0 \mu} \, \dfrac{1}{k^2 c^2} \, (-\sinh\sigma \, E'^3 + \cosh\sigma \, kcB'^2) \; , \\ \sinh\sigma \, D'^2 + \cosh\sigma \, \dfrac{H'^3}{kc} = \dfrac{1}{\mu_0 \mu} \, \dfrac{1}{k^2 c^2} \, (\sinh\sigma \, E'^2 + \cosh\sigma \, kcB'^3) \; . \end{cases}$$

They may be rewritten as

$$\begin{cases} D'^1 = \epsilon_0 \epsilon \, E'^1 \; , \qquad H'^1 = \dfrac{1}{\mu_0 \mu} \, B^{\,1} \; ; \\ \cosh\sigma \, D'^2 + \sinh\sigma \, \dfrac{H'^3}{kc} = \epsilon_0 \epsilon \, (\cosh\sigma \, E'^2 + \sinh\sigma \, kcB'^3) \; , \\ \sinh\sigma \, D'^2 + \cosh\sigma \, \dfrac{H'^3}{kc} = \dfrac{1}{\mu_0 \mu} \, \dfrac{1}{k^2 c^2} \, (\sinh\sigma \, E'^2 + \cosh\sigma \, kcB'^3) \; ; \\ \cosh\sigma \, D'^3 - \sinh\sigma \, \dfrac{H'^2}{kc} = \epsilon_0 \epsilon \, (\cosh\sigma \, E'^3 - \sinh\sigma \, kcB'^2) \; , \\ -\sinh\sigma \, D'^3 + \cosh\sigma \, \dfrac{H'^2}{kc} = \dfrac{1}{\mu_0 \mu} \, \dfrac{1}{c^2} \, (-\sinh\sigma \, E'^3 + \cosh\sigma \, kcB'^2) \; . \end{cases}$$

and further with the help of

$$\frac{1}{\mu_0 \mu k^2 c^2} = \frac{\epsilon_0 \epsilon \mu_0 \mu}{\mu_0 \mu} = \epsilon_0 \epsilon \; ,$$

[9] Everywhere instead of c, there appears kc.

they become

$$\left\{\begin{array}{l} \cosh\sigma\ D'^2 + \sinh\sigma\ \dfrac{H'^3}{kc} = \epsilon_0\epsilon\ (\cosh\sigma\ E\,'^2 + \sinh\sigma\ kcB'^3)\ , \\ \sinh\sigma\ D'^2 + \cosh\sigma\ \dfrac{H'^3}{kc} = \epsilon_0\epsilon\ (\sinh\sigma\ E'^2 + \cosh\sigma\ kcB'^3)\ , \\ \cosh\sigma\ D'^3 - \sinh\sigma\ \dfrac{H'^2}{kc} = \epsilon_0\epsilon\ (\cosh\sigma\ E'^3 - \sinh\sigma\ kcB'^2)\ , \\ -\sinh\sigma\ D'^3 + \cosh\sigma\ \dfrac{H'^2}{kc} = \epsilon_0\epsilon\ (-\sinh\sigma\ E'^3 + \cosh\sigma\ kcB'^2)\ . \end{array}\right.$$

So we get

$$D'^2 = \epsilon_0\epsilon\ E'^2\ , \qquad D'^3 = \epsilon_0\epsilon\ E'^3\ , \qquad H'^3 = \frac{1}{\mu_0\mu}\ B'^3\ , \qquad H'^2 = \frac{1}{\mu_0\mu}\ B'^2\ .$$

We have arrived at an unexpected and most attractive result: the field equations (9.1) turn out to be invariant under the *modified Lorentz transformations*. This is a significant theoretical argument in favor of the Lorentz symmetry involving the light velocity in a medium with light velocity kc instead of the light velocity in the vacuum c. Such a modified theoretical scheme looks simpler and more attractive then the commonly used one.

10 The 4-tensor formalism

The two Maxwell equations with sources

$$\text{div } \mathbf{D} = J^0\ ,\ \text{rot } \frac{\mathbf{H}}{c} = \frac{\mathbf{J}}{c} + \frac{\partial \mathbf{D}}{\partial ct} \tag{10.1}$$

can be presented in a very compact and simple form if one introduces a special notation with the use of indices taking over four values:

$$x^a = (x^0 = ct;\ x^i),\ \partial_a = \frac{\partial}{\partial x^a}\ , j^{\,a} = \left(J^0, \frac{J^i}{c}\right)\ , (H^{ab}) = \left(\begin{array}{cccc} 0 & -D^1 & -D^2 & -D^3 \\ +D^1 & 0 & -H^3/c & +H^2/c \\ +D^1 & +H^3/c & 0 & -H^1/c \\ +D^3 & -H^2/c & +H^1/c & 0 \end{array}\right) .$$

In the following, we consider the rule of changing the location (at bottom or at top) of any index-symbol:

$$A^0 = +A_0\ ,\ A^i = -A_i\ ,\ (i = 1,2,3)\ , \qquad \mathbf{A} = (A^1, A^2, A^3) = (-A_1, -A_2, -A_3)\ .$$

We further accept Einstein's convention on assumed summation over any two repeated indexes[10]: the special sign of summing $\sum$ will not be written if two identical indexes are encountered in a formula, e.g., $C^a B_a = C^0 B_0 - \mathbf{C}\ \mathbf{B}$.

Now the main assertion is that (10.1) are equivalent to the tensor one

$$\partial_b\ H^{ba} = j^a\ . \tag{10.2}$$

Indeed, in (10.2), we have:

$$a = 0 \ \Rightarrow\ \left\{\begin{array}{l} \partial_b\ H^{b0} = j^0\ ,\ \ \partial_1\ H^{10} + \partial_2\ H^{20} + \partial_3\ H^{30} = j^0\ , \\ \partial_1 D^1 + \partial_2 D^2 + \partial_3 D^3 = \rho\ , \text{div } \mathbf{D} = \rho\ . \end{array}\right.$$

[10] We use the signature $+, -, -, -$ according to Feynman.

In the same way, when $a = 1, 2, 3$:, we have, e.g.,

$$a = 1 \quad \Rightarrow \quad \begin{cases} \partial_b \; H^{b1} = j^1 \; , \quad \partial_0 \; H^{01} + \partial_2 \; H^{21} + \partial_3 \; H^{31} = j^1 \; , \\ \partial_2 \; H^3 - \partial_3 \; H^2 = \dfrac{\partial}{\partial t} D^1 + J^1 \; ; \end{cases}$$

and so on. Now let us consider the two remaining Maxwell equations

$$\text{div } c\mathbf{B} = 0 \; , \qquad \text{rot } \mathbf{E} = -\frac{\partial c\mathbf{B}}{\partial ct} \; . \tag{10.3}$$

In order to deal with these two equations, Minkowski introduced [7–11] another tensor F^{ab}:

$$(F^{ab}) = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ +E^1 & 0 & -cB^3 & +cB^2 \\ +E^1 & +cB^3 & 0 & -cB^1 \\ +E^3 & -cB^2 & +cB^1 & 0 \end{pmatrix} .$$

The main assertion here is that the remaining Maxwell equations (10.3) are equivalent to the tensor equation

$$\partial_c \; F_{ab} + \partial_a \; F_{bc} + \partial_b \; F_{ca} = 0 \; . \tag{10.4}$$

In the left side of (10.4) we have a 3-index quantity which is skew-symmetric with respect to any pair of indexes, so that in (10.4) we have only four different equations corresponding to the $\binom{4}{2} = 4$ combinations

$$(cab) = (123), \; (012), \; (023), \; (013) \; .$$

From (10.4) we can derive the following equations:

$$\begin{array}{llll} (123) : & \partial_1 F_{23} + \partial_2 F_{31} + \partial_3 F_{12} = 0 \; , & -\partial_1 B^1 - \partial_2 B^2 - \partial_3 B^3 = 0, & \text{div } \mathbf{B} = 0 \; ; \\ (012) : & \partial_0 F_{12} + \partial_1 F_{20} + \partial_2 F_{01} = 0 \; , & \partial_1 E^2 - \partial_2 E^1 = -\partial_0 B^3, & (\text{rot } \mathbf{E})^3 = -(\partial_0 \mathbf{B})^3; \end{array}$$

and so on. Thus, we have arrived at the compact tensor form of the Maxwell equations:

$$\partial_b \; H^{ba} = j^a \; , \qquad \partial_c \; F_{ab} + \partial_a \; F_{bc} + \partial_b \; F_{ca} = 0 \; ,$$

where two electromagnetic tensors are used.

The Maxwell equations in other variables (in which they exhibit symmetry under modified Lorentz transformations)

$$x^0 = kct, \qquad j^0 = \rho, \qquad \mathbf{j} = \frac{\mathbf{J}}{kc}, \qquad \mathbf{d} = \mathbf{D}, \qquad \mathbf{h} = \frac{\mathbf{H}}{kc},$$

are

$$\begin{cases} \text{div } \mathbf{d} = j^0 \; , \quad \text{rot } \mathbf{h} = \mathbf{j} + \dfrac{\partial \mathbf{d}}{\partial x^0}, \\ \text{div } \mathbf{h} = 0 \; , \quad \text{rot } \mathbf{d} = -\dfrac{\partial \mathbf{h}}{\partial x^0} \; . \end{cases}$$

and may be rewritten with the use of only one electromagnetic tensor[11]

$$(f^{AB}) = \begin{pmatrix} 0 & -d^1 & -d^2 & -d^3 \\ +d^1 & 0 & -h^3 & +h^2 \\ +d^1 & +h^3 & 0 & -h^1 \\ +d^3 & -h^2 & +h^1 & 0 \end{pmatrix}$$

[11] Here and in the following, the capital letters as tensor indexes means that such quantities transform in accordance with the modified Lorentz symmetry.

in the form of two tensor equations

$$\partial_B\, f^{BA} = j^A\,, \qquad \partial_C\, f_{AB} + \partial_A\, f_{BC} + \partial_B\, f_{CA} = 0\,.$$

Moreover, the Maxwell equations, invariant under modified Lorentz transformations, may be rewritten with the help of two tensors as well. Indeed, these equations can be written as

$$\begin{cases} \text{div}\ kc\mathbf{B} = 0\,, \quad \text{rot}\ \mathbf{E} = -\dfrac{\partial}{\partial kct}\, kc\mathbf{B}\,, \\ \text{div}\ \mathbf{D} = J^0\,, \quad \text{rot}\ \dfrac{\mathbf{H}}{kc} = \dfrac{\mathbf{J}}{kc} + \dfrac{\partial}{\partial kct}\,\mathbf{D}\,. \end{cases} \tag{10.5}$$

From here, by introducing the (modified) electromagnetic tensors:

$$(H^{AB}) = (\mathbf{D}, \mathbf{H}/kc)\,, \qquad (F^{AB}) = (\mathbf{E}, kc\mathbf{B}) \tag{10.6}$$

we note that (10.5) can be readily written as:

$$\partial_B\, H^{BA} = j^A\,, \qquad \partial_C\, F_{AB} + \partial_A\, F_{BC} + \partial_B\, F_{CA} = 0\,.$$

11 Ordinary and modified 4-vectors and tensors

The main significant property of the notation introduced by Poincaré-Minkowski is that it allows us to determine in all details the correct Lorentz formulas for any physical quantity. The recipe is easy: *the Lorentz formulas for any quantity are determined in a straightforward manner by its tensorial nature.*

Consider several examples. Any physical entity with one upper index, first-order tensor, behaves like the coordinate 4-vector x^a:

$$\begin{aligned} x'^0 &= \cosh\beta\, x^0 - \sinh\beta\, x^1\,, \\ x'^1 &= -\sinh\beta\, x^0 + \cosh\beta\, x^1\,, \\ x'^2 &= x^2\,, \qquad x'^3 = x^3\,, \end{aligned}$$

i.e., 1-rank tensor A^a transforms as follows

$$\begin{aligned} A'^0 &= \cosh\beta\, A^0 - \sinh\beta\, A^1\,, \\ A'^1 &= -\sinh\beta\, A^0 + \cosh\beta\, A^1\,, \\ A'^2 &= A^2\,, \qquad A'^3 = A^3\,. \end{aligned}$$

The same situation, but in matrix form, looks as $A'^a = L'^a{}_b(\beta)\ A^b$. In accordance with the above assertion, a 2-rank tensor will be translated by the formulas

$$K'^{ab} = L^a{}_m(\beta) L^b{}_n(\beta)\ K^{mn}\,.$$

It can be readily verified that all the transformation formulas used above coincide the formulas following from the general tensor formalism. When considering modified Lorentz symmetry, the whole the tensor technique remains the same, except of small alterations, like : $x'^B = L'^B{}_C(\sigma)\ x^C$, etc.

12 Minkowski's relations in covariant tensor form

We consider Minkowski's equations generated by a special displacement of the reference frame along the axis x:

$$D^i = \epsilon_0\epsilon\ E^i \qquad \Longrightarrow \qquad \begin{cases} D'^1 = \epsilon_0\epsilon\ E'^1\ , \\ \cosh\beta\ D'^2 + \sinh\beta\ \dfrac{H'^3}{c} = \epsilon_0\epsilon\ (\cosh\beta\ E'^2 + \sinh\beta\ cB'^3)\ , \\ \cosh\beta\ D'^3 - \sinh\beta\ \dfrac{H'^2}{c} = \epsilon_0\epsilon\ (\cosh\beta\ E'^3 - \sinh\beta\ cB'^2)\ ; \end{cases}$$

$$H^i = \frac{1}{\mu_0\mu}\ B^i \qquad \Longrightarrow \qquad \begin{cases} H'^1 = \dfrac{1}{\mu_0\mu}\ B^1\ , \\ -\sinh\beta\ D'^3 + \cosh\beta\ \dfrac{H'^2}{c} = \dfrac{1}{\mu_0\mu}\ \dfrac{1}{c^2}\ (-\sinh\beta\ E'^3 + \cosh\beta\ cB'^2)\ , \\ \sinh\beta\ D'^2 + \cosh\beta\ \dfrac{H'^3}{c} = \dfrac{1}{\mu_0\mu}\ \dfrac{1}{c^2}\ (\sinh\beta\ E'^2 + \cosh\beta\ cB'^3) \end{cases}$$

may be quite easily rewritten in a special form which remains unchanged under Lorentz transformations, including arbitrary rotations and (uniform) motion.

The trick which we shall use below is simple but useful and often applicable. It is based on the following property of the tensor formalism: *if we think (know) that a certain physical equation must be Lorentz invariant and an explicit form of the equation is given only in some particular reference frame then its invariant form may be found with the help of Lorentz transformations. The same may be achieved if we can derive from the particular equation its general tensor form.*

One special notion, 4-vector of velocity, is needed for the following. It may be introduced using the following simple formal considerations. Let a material particle move steadily in some inertial reference frame:

$$x^0 = ct\ , \qquad x^i = v^i\ t\ .$$

Since x^a is a 4-vector, its differential dx^a will be a 4-vector too; moreover, we have a scalar quantity (which is called an *interval*) with respect to the Lorentz group transformations:

$$s^2 = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2\ ,$$

$$ds^2 = (dx^0)^2 - dl^2\ ,\ ds = c\ dt\ \sqrt{1 - \frac{v^2}{c^2}}\ .$$

Dividing dx^a by ds, we get the velocity 4-vector:

$$u^a = \frac{dx^a}{ds} = \frac{(cdt, dx^1, dx^2, dx^3)}{cdt\ \sqrt{1 - v^2/c^2}} = \left(\frac{1}{\sqrt{1 - v^2/c^2}}, \frac{v^i/c}{\sqrt{1 - v^2/c^2}} \right)\ .$$

Now we are ready to obtain the tensor form of Minkowski equations. To this end, we take a particular 4-velocity vector:

$$u^a = \left(\frac{1}{\sqrt{1 - v^2/c^2}},\ \frac{-v/c}{\sqrt{1 - v^2/c^2}},\ 0,\ 0 \right) = (\cosh\beta, -\sinh\beta, 0, 0)\ . \qquad (12.1)$$

The first tensor relation we need is [1]

$$H^{ab}u_b = \epsilon_0\epsilon\ F^{ab}u_b\ . \qquad (12.2)$$

When u^a is given by (12.1), (12.2) will take the form

$$
\begin{aligned}
a=0 \;\Rightarrow\quad & H^{01}u_1 = \epsilon_0\epsilon\; F^{01}u_1\;,\; D^1\;\sinh\beta = \epsilon_0\epsilon\;\sinh\beta E^1\;,\; D^1 = \epsilon_0\epsilon E^1\;;\\
a=1 \;\Rightarrow\quad & H^{10}u_0 = \epsilon_0\epsilon\; F^{10}u_0\;, -D^1\;\cosh\beta = -\epsilon_0\epsilon\;\cosh\beta E^1\;,\; D^1 = \epsilon_0\epsilon E^1\;;\\
a=2 \;\Rightarrow\quad & H^{20}u_0 + H^{21}u_1 = \epsilon_0\epsilon\;(F^{20}u_0 + F^{21}u_1)\;,\\
& D^2\cosh\beta + \frac{H^3}{c}\;\sinh\beta = \epsilon_0\epsilon\;(E^2\cosh\beta + cB^3\;\sinh\beta)\;;\\
a=3 \;\Rightarrow\quad & H^{30}u_0 + H^{31}u_1 = \epsilon_0\epsilon\;(F^{30}u_0 + F^{31}u_1)\;,\\
& D^3\cosh\beta - \frac{H^2}{c}\;\sinh\beta = \epsilon_0\epsilon\;(E^3\cosh\beta - cB^2\;\sinh\beta)\;.
\end{aligned}
$$

Therefore, (12.2) is equivalent to (9.2). The second tensor relation we need is ([1])

$$
(H^{ab}u^c + H^{bc}u^a + H^{ca}u^b) = \frac{1}{c^2\mu\mu_0}\;(F^{ab}u^c + F^{bc}u^a + F^{ca}u^b)\;.
$$

With u^a as in (12.1), it gives

$$
\underline{(123)}\;,\qquad (H^{12}u^3 + H^{23}u^1 + H^{31}u^2) = \frac{1}{c^2\mu\mu_0}\;(F^{12}u^3 + F^{23}u^1 + F^{31}u^2),\qquad H^1 = \frac{1}{\mu\mu_0}\;B^1\;;
$$

$$
\underline{(023)}\;,\qquad (H^{02}u^3 + H^{23}u^0 + H^{30}u^2) = \frac{1}{c^2\mu\mu_0}\;(F^{02}u^3 + F^{23}u^0 + F^{30}u^2)\;,\qquad H^1 = \frac{1}{\mu\mu_0}\;B^1\;;
$$

$$
\begin{aligned}
\underline{(012)}\;,\qquad & (H^{01}u^2 + H^{12}u^0 + H^{20}u^1) = \frac{1}{c^2\mu\mu_0}\;(F^{01}u^2 + F^{12}u^0 + F^{20}u^1)\;,\\
& \left(\frac{H^3}{c}\;\cosh\beta + D^2\;\sinh\beta\right) = \frac{1}{c^2\mu\mu_0}\;(cB^3\;\cosh\beta + E^2\;\sinh\beta)\;;
\end{aligned}
$$

$$
\begin{aligned}
\underline{(013)}\;,\qquad & (H^{01}u^3 + H^{13}u^0 + H^{30}u^1) = \frac{1}{c^2\mu\mu_0}\;(F^{01}u^3 + F^{13}u^0 + F^{30}u^1)\;,\\
& \left(\frac{H^2}{c}\;\cosh\beta - D^3\;\sinh\beta\right) = \frac{1}{c^2\mu\mu_0}\;(cB^2\;\cosh\beta - E^3\;\sinh\beta)\;,
\end{aligned}
$$

and these coincide with relations (9.3).

Thus, all the six Minkowski equations are equivalent to the tensor ones:

$$
H^{ab}u_b = \epsilon_0\epsilon F^{ab}u_b\;, \tag{12.3}
$$

$$
H^{ab}u^c + H^{bc}u^a + H^{ca}u^b = \frac{1}{c^2\mu\mu_0}\;(F^{ab}u^c + F^{bc}u^a + F^{ca}u^b)\;. \tag{12.4}
$$

Although we have verified them only with the use of a special Lorentz transformation, but the tensor form itself guarantees that they will be correct for any arbitrary Lorentz one[12].

In the vacuum case, when $\epsilon = 1, \mu = 1$, the equations (12.3) and (12.4) may be rewritten differently

$$
H^{ab}u_b = \epsilon_0 F^{ab}u_b\;, \tag{12.5}
$$

$$
H^{ab}u^c + H^{bc}u^a + H^{ca}u^b = \epsilon_0(F^{ab}u^c + F^{bc}u^a + F^{ca}u^b)\;. \tag{12.6}
$$

[12] To be exact, we must recognize that in this statement there is presented an additional element of definition.

These tensor equations admit a simple solution. Indeed, let us multiply (12.6) by u_c (considering $u^c u_c = +1$). Then

$$H^{ab} = \epsilon_0 (F^{ab} + F^{bc} u_c u^a + F^{ca} u_c u^b) - H^{bc} u_c u^a - H^{ca} u_c u^b \ ,$$

from where, having in view (12.5), we infer

$$H^{ab} = \epsilon_0 F^{ab} \ .$$

This tensor condition in component form is expressed by six relations (the same ones which were earlier obtained in (9.6))

$$D^i = \epsilon_0 \ E^i \ , \qquad H^i = \frac{1}{\mu_0} \ B^i \ .$$

However, in the case of the medium, similar calculation leads to a very different result. Indeed, let us multiply (12.4) by u_c:

$$H^{ab} + H^{bc} u_c u^a + H^{ca} u_c u^b = \frac{1}{c^2 \mu \mu_0} \ (F^{ab} + F^{bc} u_c u^a + F^{ca} u_c u^b) \ .$$

From this, taking (12.3), we get

$$H^{ab} = \epsilon_0 \epsilon k^2 \ F^{ab} + \epsilon_0 \epsilon \ (k^2 - 1) \ \Big(\ F^{bc} u_c u^a - F^{ac} u_c u^b \ \Big) \ . \tag{12.7}$$

Evidently, these form the earlier found covariant tensor from (9.5).

Take notice that while using the Maxwell theory with only one (modified) tensor f^{AB}

$$\partial_B \ f^{BA} = j^A \ , \ \partial_C \ f_{AB} + \partial_A \ f_{BC} + \partial_B \ f_{CA} = 0,$$

no additional condition between electromagnetic tensors is needed at all. Some clarifying analysis can be easily done. In this case, there arise modified Minkowski relations (c is replaced by kc):

$$D^i = \epsilon_0 \epsilon \ E^i \ \Rightarrow \ \begin{cases} D'^1 = \epsilon_0 \epsilon \ E'^1 \ , \\ \cosh\sigma \ D'^2 + \sinh\sigma \ \dfrac{H'^3}{kc} = \epsilon_0 \epsilon \ (\cosh\sigma \ E'^2 + \sinh\sigma \ kcB'^3) \ , \\ \cosh\sigma \ D'^3 - \sinh\sigma \ \dfrac{H'^2}{kc} = \epsilon_0 \epsilon \ (\cosh\sigma \ E'^3 - \sinh\sigma \ kcB'^2) \ , \end{cases} \tag{12.8}$$

$$H^i = \frac{1}{\mu_0 \mu} \ B^i \ \Rightarrow \ \begin{cases} H'^1 = \dfrac{1}{\mu_0 \mu} \ B^1 \ , \\ -\sinh\sigma \ D'^3 + \cosh\sigma \ \dfrac{H'^2}{kc} = \dfrac{1}{\sigma_0 \mu} \ \dfrac{1}{k^2 c^2} \ (-\sinh\sigma \ E'^3 + \cosh\sigma \ kcB'^2) \ , \\ \sinh\sigma \ D'^2 + \cosh\sigma \ \dfrac{H'^3}{kc} = \dfrac{1}{\mu_0 \mu} \ \dfrac{1}{k^2 c^2} \ (\sinh\sigma \ E'^2 + \cosh\sigma \ kcB'^3) \ . \end{cases} \tag{12.9}$$

Now a modified 4-velocity U^A is needed:

$$U^A = \frac{dx^a}{ds} = \frac{(kcdt, dx^1, dx^2, dx^3)}{kcdt \ \sqrt{1 - v^2/k^2c^2}} = \left(\frac{1}{\sqrt{1 - v^2/k^2c^2}}, \frac{v^i/kc}{\sqrt{1 - v^2/k^2c^2}} \right) \ ,$$

and its particular form

$$U^A = \left(\frac{1}{\sqrt{1 - v^2/k^2c^2}}, \frac{-v/kc}{\sqrt{1 - v^2/k^2c^2}}, \ 0, \ 0 \right) = (\cosh\sigma, -\sinh\sigma, 0, 0) \ .$$

According to (10.6), the tensor representation of (12.8) and (12.9) is:

$$H^{AB}U_B = \epsilon_0\epsilon F^{AB}U_B \, ,$$
$$H^{AB}U^C + H^{BC}U^A + H^{CA}U^B = \frac{1}{k^2c^2\mu\mu_0}(F^{AB}U^C + F^{BC}U^A + F^{CA}U^B) \, . \tag{12.10}$$

The latter equation may be rewritten as

$$H^{AB}U^C + H^{BC}U^A + H^{CA}U^B = \epsilon_0\epsilon \, (F^{AB}U^C + F^{BC}U^A + F^{CA}U^B) \, .$$

Multiplying it by U_C:

$$H^{AB} + H^{BC}U_CU^A + H^{CA}U_CU^B = \epsilon_0\epsilon \, (F^{AB} + F^{BC}U_CU^A + F^{CA}U_CU^B) \, ,$$

from where, using (12.10), it follows

$$H^{AB} = \epsilon_0\epsilon \; F^{AB} \, .$$

which, in components, looks as claimed

$$D^i = \epsilon_0\epsilon \; E^i \, , \qquad H^i = \frac{1}{\mu_0\mu} \; B^i \, .$$

13 Potentials in a medium

In remains to see which peculiarities arise from the presence of a uniform medium, when we try do describe electromagnetic fields in terms of *the scalar and vector potentials* $\varphi, \mathbf{A}$:

$$\{ \, \mathbf{E}, \, \mathbf{B}, \, \mathbf{D}, \, \mathbf{H} \, \} \implies \qquad \{ \, \varphi, \, \mathbf{A} \, \} \, .$$

We might anticipate some difficulties because the Maxwell theory in the medium exhibits symmetry under ordinary (the vacuum light velocity based) Lorentz transformations, *only if the two electromagnetic tensors, H^{ab} and F^{ab}, are used.*

However, there is no ground for feeling enthusiastic about the existence of the relativistically invariant formulas (12.3) and (12.4):

$$H^{ab}u_b = \epsilon_0\epsilon F^{ab}u_b \, ,$$
$$H^{ab}u^c + H^{bc}u^a + H^{ca}u^b = \frac{1}{c^2\mu\mu_0}(F^{ab}u^c + F^{bc}u^a + F^{ca}u^b) \, .$$

These formulas involve the 4-velocity vector, which characterizes the motion of the medium under the inertial reference frame. In other words, these formulas show an explicit dependence of the basic electrodynamic equations on the absolute velocity of a moving body. This contradicts the initial principles and the main claims of Special Relativity Theory.

Let us write down the Maxwell equations again in operator form:

$$\nabla \bullet \; \mathbf{B} = 0 \, , \qquad \nabla \times \; \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \, , \tag{13.1}$$

$$\nabla \bullet \; \mathbf{E} = \frac{1}{\epsilon\epsilon_0}\rho \,', \qquad \frac{1}{\mu\mu_0}\nabla \times \; \mathbf{B} = \mathbf{J} + \; \epsilon\epsilon_0\frac{\partial \mathbf{E}}{\partial t}. \tag{13.2}$$

The most general substitution for potentials φ, $\mathbf{A}$ which transforms the first two equations (13.1) into identities, has the form

$$\mathbf{B} = d \, \nabla \times \mathbf{A}, \;\; \mathbf{E} = -n \, \nabla\varphi - d \, \frac{\partial \mathbf{A}}{\partial t} \;\; , \tag{13.3}$$

where d, n are some yet unknown parameters. The first equation in (13.2) yields

$$\left(-n\nabla^2\varphi + dm\frac{\partial^2\varphi}{\partial t^2}\right) = \frac{1}{\epsilon\epsilon_0}\rho + d\frac{\partial}{\partial t}\left(\nabla\bullet\mathbf{A} + m\frac{\partial\varphi}{\partial t}\right),$$

or

$$\left(-\nabla^2\varphi + \frac{dm}{n}\frac{\partial^2\varphi}{\partial t^2}\right) = \frac{1}{n\epsilon\epsilon_0}\rho + \frac{d}{n}\frac{\partial}{\partial t}\left(\nabla\bullet\mathbf{A} + m\frac{\partial\varphi}{\partial t}\right), \tag{13.4}$$

The second equations in (13.2) lead to

$$\frac{d}{\mu\mu_0}\nabla\times(\nabla\times\mathbf{A}) = \mathbf{J} + \epsilon\epsilon_0\frac{\partial}{\partial t}\left(-n\nabla\varphi - d\frac{\partial\mathbf{A}}{\partial t}\right),$$

or

$$\nabla\times\ (\nabla\times\mathbf{A}) = \frac{\mu\mu_0}{d}\,\mathbf{J} + \frac{\epsilon\epsilon_0\mu\mu_0}{d}\,\frac{\partial}{\partial t}\left(-n\,\nabla\varphi - d\,\frac{\partial\mathbf{A}}{\partial t}\right).$$

From this, with the help of the identity

$$\nabla\times(\nabla\times\mathbf{A}) = -\nabla^2\,\mathbf{A} + \nabla\,(\nabla\bullet\mathbf{A})\,,$$

we get

$$-\nabla^2\,\mathbf{A} + \nabla\,(\nabla\bullet\mathbf{A}) = \frac{\mu\mu_0}{d}\,\mathbf{J} + \frac{\epsilon_0\epsilon\mu_0\mu}{d}\,\frac{\partial}{\partial t}\left(-n\,\nabla\varphi - d\,\frac{\partial\mathbf{A}}{\partial t}\right),$$

or

$$\left(-\nabla^2\,\mathbf{A} + \epsilon_0\epsilon\mu_0\mu\,\frac{\partial^2}{\partial t^2}\,\mathbf{A}\right) = \frac{\mu\mu_0}{d}\,\mathbf{J} - \nabla\left(\nabla\bullet\mathbf{A} + \frac{\epsilon_0\epsilon\mu_0\mu\,n}{d}\,\frac{\partial\varphi}{\partial t}\right). \tag{13.5}$$

Comparing (13.4) and (13.5), we see that it suffices to impose the equalities

$$\frac{dm}{n} = \epsilon_0\epsilon\mu_0\mu\,, \quad \text{or, equivalently,} \quad m = \frac{\epsilon_0\epsilon\mu_0\mu\,n}{d}$$

so that (13.4) and (13.5) will have quite symmetrical form with the same wave operator on the left hand side:

$$\left(-\nabla^2\varphi + \epsilon_0\epsilon\mu_0\mu\,\frac{\partial^2\varphi}{\partial t^2}\right) = \frac{1}{n\,\epsilon\epsilon_0}\,\rho + \frac{d}{n}\,\frac{\partial}{\partial t}\left(\nabla\bullet\mathbf{A} + m\frac{\partial\varphi}{\partial t}\right),$$

$$\left(-\nabla^2\,\mathbf{A} + \epsilon_0\epsilon\mu_0\mu\,\frac{\partial^2\mathbf{A}}{\partial t^2}\right) = \frac{\mu\mu_0}{d}\,\mathbf{J} - \nabla\left(\nabla\bullet\mathbf{A} + m\,\frac{\partial\varphi}{\partial t}\right).$$

With the use of the coonstants c and k, defined by:

$$c^2 = \frac{1}{\epsilon_0\mu_0}, \qquad k^2 = \frac{1}{\epsilon\mu},$$

the previous equations become

$$\frac{1}{k^2c^2} = \frac{dm}{n}\,, \tag{13.6}$$

$$\left(-\nabla^2\varphi + \frac{1}{k^2c^2}\,\frac{\partial^2\varphi}{\partial t^2}\right) = \frac{1}{n\,\epsilon\epsilon_0}\,\rho + \frac{d}{n}\,\frac{\partial}{\partial t}\left(\nabla\bullet\mathbf{A} + m\frac{\partial\varphi}{\partial t}\right), \tag{13.7}$$

$$\left(-\nabla^2\,\mathbf{A} + \frac{1}{k^2c^2}\,\frac{\partial^2\mathbf{A}}{\partial t^2}\right) = \frac{\mu\mu_0}{d}\,\mathbf{J} - \nabla\left(\nabla\bullet\mathbf{A} + m\,\frac{\partial\varphi}{\partial t}\right). \tag{13.8}$$

The relation (13.6) admits several distinct solutions. The most symmetrical and simplifying all formulas substitution seems to be

$$m = \frac{1}{kc}\ , \quad n = \frac{1}{\epsilon_0 \epsilon}\ , \text{ whence } d = \frac{n}{kc} = \frac{1}{kc\ \epsilon_0\epsilon}\ , \quad \frac{\mu\mu_0}{d} = \mu\mu_0\epsilon\epsilon_0\ kc = \frac{1}{kc}\ .$$

Then (13.7) and (13.8) lead to

$$\begin{aligned} &\left(-\nabla^2\varphi + \frac{1}{k^2c^2}\frac{\partial^2\varphi}{\partial t^2}\right) = \rho + \frac{1}{kc}\frac{\partial}{\partial t}\left(\nabla \bullet \mathbf{A} + \frac{1}{kc}\frac{\partial\varphi}{\partial t}\right), \\ &\left(-\nabla^2\mathbf{A} + \frac{1}{k^2c^2}\frac{\partial^2\mathbf{A}}{\partial t^2}\right) = \frac{\mathbf{J}}{kc} - \nabla\left(\nabla \bullet \mathbf{A} + \frac{1}{kc}\frac{\partial\varphi}{\partial t}\right). \end{aligned} \tag{13.9}$$

Moreover, (13.9) may be rewritten as a (modified) tensor equation:

$$\partial^B\partial_B\ A^C = j^C + \partial^C(\partial_B A^B)\ , \tag{13.10}$$

where

$$x^C = (kct, x^i)\ ,\ A^C = (\varphi, A^i),\ j^C = (\rho,\ \frac{J^i}{kc})\ .$$

Then (13.10) evidently shows its invariance under modified Lorentz transformations constructed on the base of the light velocity kc in the medium. The initial relations (13.3), which introduced the electromagnetic potentials lead to the following relations

$$\begin{aligned} &\mathbf{B} = d\ \nabla\times\mathbf{A} = \frac{1}{\epsilon\epsilon_0 kc}\ \nabla\times\mathbf{A} \quad \Rightarrow\ \nabla\times\mathbf{A} = \frac{\mathbf{B}}{\mu_0\mu\ kc} = \mathbf{h}\ ; \\ &\mathbf{E} = \frac{1}{\epsilon_0\epsilon}\ \nabla\varphi - \frac{1}{\epsilon\epsilon_0 kc}\ \frac{\partial\mathbf{A}}{\partial t} \qquad \Rightarrow\ -\nabla\varphi - \frac{1}{kc}\ \frac{\partial\mathbf{A}}{\partial t} = \epsilon_0\epsilon\ \mathbf{E} = \mathbf{d} \end{aligned}$$

may be readily translated to tensor form:

$$f_{BC} = \partial_B A_C - \partial_C A_B\ , \tag{13.11}$$

where f_{ab} is the electromagnetic tensor for Maxwell equations in modified variables:

$$x^0 = kct\ ,\ j^0 = \rho\ ,\ \mathbf{j} = \frac{\mathbf{J}}{kc}\ , \qquad \mathbf{d} = \mathbf{D},\ \mathbf{h} = \frac{\mathbf{H}}{kc}\ ,\ \ f^{AB} = (\mathbf{d}, \mathbf{h})$$

namely

$$\text{div}\ \mathbf{d} = j^0\ ,\ \text{rot}\ \mathbf{h} = \mathbf{j}\ +\ \frac{\partial}{\partial x^0}\ \mathbf{d}\ .$$

The tensorial relation (13.11) easily reveals a gauge freedom in determining of electromagnetic potentials:

$$A'_B = A_B + \partial_B\Lambda \ \ \Rightarrow\ f'_{BC} = f_{BC}\ .$$

It was mentioned above that often the Lorentz gauge condition is taken to be the most convenient as

$$\partial_B A^B = 0\ .$$

In this respect, there may be formulated two problems.

Problem 1.

Let a given potential A^B not obey the Lorentz condition; then a gauge transformation Λ leads us to a new potential (A'^B satisfying this condition, i.e.,

$$\partial_B A'^B = 0.$$

The solution to this problem can be considered of the form $A'^B = \partial^B \Lambda + A^B$, which yields the wave PDE:

$$\partial_B \partial^B \, \Lambda = -\partial_B A^B \ ,$$

with the unknown Λ and with non-zero right-hand side which is given through the old potential A^B.

Problem 2.

> Let A^B be a potential satisfying Lorentz condition; which are the gauge transformations preserving the Lorentz gauge relationship.

From the invariance of the Lorentz condition

$$\partial_B A^B = 0 \ , \qquad \partial_B A'^B = 0 \ ,$$

it follows

$$\partial_B \partial^B \, \Lambda = 0 \ , \tag{13.12}$$

i.e., every solution Λ of the wave equation (13.12) preserves the Lorentz gauge.

Thus, as consequence of the change of variables, we infer:

$$(\mathbf{E}, \mathbf{D}, \mathbf{B}, \mathbf{H}) \implies (\varphi, \mathbf{A}) = A^C.$$

Simplicity has been achieved: all the Maxwell electrodynamics is formally equivalent to one single equation for 1-order tensor A^B. In the Lorentz gauge, the Maxwell electrodynamics looks most simple and beautiful: namely it reduces to the wave equation:

$$\partial^B \partial_B \, A^C = j^C \ , \qquad \partial_B A^B = 0 \ . \tag{13.13}$$

One should note again that to the solutions of equation (13.13) there correspond wave processes propagating with the velocity of the light *in the medium, not in the vacuum.*

The choice of solution in (13.6) that was taken above is not unique. Equally, starting from (13.6),(13.7),(13.8), we may chose the more traditional solutions:

$$\begin{cases} d = 1, \qquad n = 1, \qquad m = \dfrac{1}{k^2 c^2} \ , \\ \left(-\nabla^2 \varphi + \dfrac{1}{k^2 c^2} \dfrac{\partial^2 \varphi}{\partial t^2} \right) = \dfrac{1}{\epsilon \epsilon_0} \, \rho \ + \dfrac{\partial}{\partial t} \left(\nabla \bullet \mathbf{A} + \dfrac{1}{k^2 c^2} \dfrac{\partial \varphi}{\partial t} \right), \\ \left(-\nabla^2 \, \mathbf{A} + \dfrac{1}{k^2 c^2} \dfrac{\partial^2 \mathbf{A}}{\partial t^2} \right) = \mu \mu_0 \, \mathbf{J} - \nabla \left(\nabla \bullet \mathbf{A} + \dfrac{1}{k^2 c^2} \dfrac{\partial \varphi}{\partial t} \right) \ . \end{cases}$$

which coincide with the ones used in the the known handbook on electrodynamics by Stratton [17]. Evidently, the two variants are totally equivalent, because they differ only in determining the units for potentials:

$$\mathbf{B} = d \, \nabla \times \mathbf{A}, \ \mathbf{E} = -n \, \nabla \varphi - d \, \frac{\partial \mathbf{A}}{\partial t}.$$

14 Potentials in a medium – ordinary approach

Now we consider an alternative way of introducing potentials into electrodynamics, in presence of a medium which has its origin in earlier investigations by Minkowski [7–11][13].

[13] Just this variant is being used mainly; see for instance the thorough review [1].

The most noticeable feature of this method consists in the following: in this approach we are able to formulate the Maxwell electrodynamics in a medium in terms of potentials with the use of the ordinary Lorentz symmetry based on the vacuum light velocity c only. Concurrently, the mathematical equations achieved look more complicated, and also these equations explicitly involve the velocity of the medium under the reference frame. The latter might be considered as return to prehistory of Special Relativity with all the search of some absolute velocities. However, we are not going to be submerged in such metaphysical subtleties. Nevertheless, one issue should be emphasized: *there exist two alternative ways to develop potential approach for electrodynamics in medium – one developed in the previous Section, and another exposed below. The tho ways are completely equivalent in mathematical sense. The first is much more simpler technically but it presumes invariance under modified Lorentz symmetry based on the light velocity kc. There arises the question, if the more complicated technique is more likely to be used only because of its concomitant treatment of ordinary Lorentz symmetry.*

Minkowski's equations in covariant tensor form are:

$$\begin{cases} H^{ab}u_b = \epsilon_0\epsilon F^{ab}u_b \ , \\ H^{ab}u^c + H^{bc}u^a + H^{ca}u^b = \dfrac{F^{ab}u^c + F^{bc}u^a + F^{ca}u^b}{c^2\mu\mu_0} \ . \end{cases}$$

It was shown in (12.7) that these are equivalent to

$$H^{ab} = \epsilon_0\epsilon k^2 \ F^{ab} + \epsilon_0\epsilon \ (k^2-1) \ (\ F^{bc}u_c u^a - F^{ac}u_c u^b \ ) \ ,$$

or

$$H_{ab} = [\ \epsilon_0\epsilon k^2 \ g_{am} \ g_{bn} + \epsilon_0\epsilon \ (k^2-1) \ u_n(g_{bm}u_a - g_{am}u_b) \ ] \ F^{mn} \ , \tag{14.1}$$

which can be rewritten as:

$$H_{ab} = \Delta_{abmn} \ F^{mn} \ , \qquad \text{where} \qquad \Delta_{abmn} = \epsilon_0\epsilon k^2 \ g_{am} \ g_{bn} + \epsilon_0\epsilon(k^2-1) \ u_n(g_{bm}u_a - g_{am}u_b) \ . \tag{14.2}$$

Firstly, the 4-order tensor connecting H^{ab} and F^{ab} was introduced (for a more general case of anisotropic medium) by Tamm and Mandel'stam [6,23,24]. For a uniform medium, according to Watson-Yauch-Riazanov [4,16] the tensor Δ_{abmn} (14.2) may be taken in another form, leading to:

$$H_{ab} = \Delta_{abmn} \ F^{mn} \ , \qquad \text{where} \qquad \Delta_{abmn} = A \ (g_{am} + Bu_a u_m) \ (g_{bn} + Bu_b u_n) \ . \tag{14.3}$$

Although this Δ_{abmn} contains a term of fourth order in velocity, only terms of second order give non-zero contributions into the formula (14.3). Let us demonstrate that (14.2) and (14.3) are the same in different forms for certain given A and B. From (14.3) it follows

$$\begin{aligned} H_{ab} &= A \ g_{am} \ g_{bn} \ F^{mn} + AB \ g_{am}Bu_b u_n \ F^{mn} + AB \ u_a u_m g_{bn} \ F^{mn} \\ &= A \ g_{am} \ g_{bn} \ F^{mn} + AB \ u_n \ (g_{am}u_b - g_{bm}u_a) \ F^{mn} \ . \end{aligned}$$

Comparing this with (14.2), we get

$$A = \epsilon_0\epsilon k^2 \ , \qquad -AB = \epsilon_0\epsilon(k^2-1) \ ,$$

from where it follows

$$A = \epsilon_0\epsilon k^2, \qquad B = \frac{1-k^2}{k^2} = \epsilon\mu - 1 \ .$$

Therefore, equations (14.3) become

$$H_{ab} = \Delta_{abmn}\, F^{mn} \;, \qquad \text{where} \qquad \Delta_{abmn} = \epsilon_0 \epsilon k^2 [\; g_{am} + (\epsilon\mu - 1)\; u_a u_m \;]\; [\; g_{bn} + (\epsilon\mu - 1)\; u_b u_n \;].$$

This very representation for 4-rank tensor relating H^{ab} to F^{ab} in a uniform medium is given in the review [1].

Now we are ready to introduce potentials. The Maxwell equations are

$$\partial_a\, H^{ab} = j^b \;, \;\; \partial_a\, (\Delta^{abmn} F_{mn}) = j^b \;, \tag{14.4}$$

$$\partial_c\, F_{ab} + \partial_a\, F_{bc} + \partial_b\, F_{ca} = 0 \;. \tag{14.5}$$

The potentials A_b are defined in such a way that equations (14.5) turn into identities:

$$F_{ab} = \partial_a A_b - \partial_b A_a \;;$$

With the help of (14.1), H^{ab} may be rewritten as

$$H^{ab} = \epsilon_0 \epsilon k^2\; (\partial^a A^b - \partial^b A^a) + \epsilon_0 \epsilon\; (k^2 - 1)(\; u^a \partial^b - u^b \partial^a\;)\; (u^n A_n)\;.$$

Therefore, (14.4) leads us to

$$\epsilon_0 \epsilon k^2\; \partial_a (\partial^a A^b - \partial^b A^a) + \epsilon_0 \epsilon (k^2 - 1) \partial_a (u^a \partial^b - u^b \partial^a)(u^n A_n) = j^b. \tag{14.6}$$

This is the main equation for electromagnetic 4-potentials in a medium, invariant under the ordinary Lorentz transformations based on the vacuum light velocity. For purely vacuum case, the factor $(k^2 - 1)$ equals to zero and (14.6) takes the more familiar form

$$\partial_a \partial^a A^b - \partial^b (\partial_a A^a) = \mu_0\; j^b \;. \tag{14.7}$$

So when we take into account the presence of a medium, the equation (14.7) extends to (14.6).

15 Geometrical modeling of the uniform media, transition the Dirac equation in the medium;

It is well-known the possibility to simulate some media in electrodynamics with the help of Riemannian geometry (see the recent consideration and bibliography in []). In this context let us consider one special form of the metrical tensor:

$$g_{\alpha\beta}(x) = \left(\begin{array}{cccc} a^2 & 0 & 0 & 0 \\ 0 & -b^2 & 0 & 0 \\ 0 & 0 & -b^2 & 0 \\ 0 & 0 & 0 & -b^2 \end{array} \right) \;, \; g^{\alpha\beta}(x) = \left(\begin{array}{cccc} a^{-2} & 0 & 0 & 0 \\ 0 & -b^{-2} & 0 & 0 \\ 0 & 0 & -b^{-2} & 0 \\ 0 & 0 & 0 & -b^{-2} \end{array} \right) \;, \tag{15.1}$$

where a^2 and b^2 are arbitrary (positive) numerical parameters. The material equations generated by this geometry are

$$D^i = \epsilon^{ik}\; E_k \;, \qquad (\epsilon^{ik}) = \epsilon_0 \sqrt{-g}\; g^{00} \left(\begin{array}{ccc} g^{11} & 0 & 0 \\ 0 & g^{22} & 0 \\ 0 & 0 & g^{33} \end{array} \right) = \epsilon_0 \frac{b}{a} \left(\begin{array}{ccc} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{array} \right) \;,$$

$$H^i = \mu^{ik}\; B_k \;, \qquad (\mu^{ik}) = \frac{1}{\mu_0} \sqrt{-g} \left(\begin{array}{ccc} g^{22} g^{33} & 0 & 0 \\ 0 & g^{33} g^{11} & 0 \\ 0 & 0 & g^{11} g^{22} \end{array} \right) = \frac{1}{\mu_0} \frac{a}{b} \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right) \;,$$

or, differently,

$$D^i = -\epsilon_0 \frac{b}{a} E_i , \qquad H^i = \frac{1}{\mu_0} \frac{a}{b} B_i , \tag{15.2}$$

and the Maxwell equations take the form

$$\begin{array}{l}
\partial_1 B_1 + \partial_2 B_2 + \partial_3 B_3 = 0 , \qquad \partial_t B_1 = \partial_2 E_3 - \partial_3 E_2 , \\
\partial_t B_2 = \partial_3 E_1 - \partial_1 E_3 , \qquad \partial_t B_3 = \partial_1 E_2 - \partial_2 E_1 , \\
\partial_i D^i = \rho , \\
\frac{\partial}{\partial x^2} H^3 - \frac{\partial}{\partial x^3} H^2 = J^1 + \frac{\partial}{\partial t} D^1 , \\
\frac{\partial}{\partial x^3} H^1 - \frac{\partial}{\partial x^1} H^3 = J^2 + \frac{\partial}{\partial t} D^2 , \\
\frac{\partial}{\partial x^1} H^2 - \frac{\partial}{\partial x^2} H^1 = J^3 + \frac{\partial}{\partial t} D^3 .
\end{array}$$

In vector terms, we get

$$\mathbf{B} = (B_i) , \qquad \mathbf{E} = (-E_i) , \qquad \mathbf{H} = (H^i) , \qquad \mathbf{D} = (D^i) ,$$

which can be rewritten as follows:

$$\left\{ \begin{array}{l}
\text{div } \mathbf{B} = 0 , \quad \text{rot } \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t} , \\
\text{div } \mathbf{D} = \rho , \quad \text{rot } \mathbf{D} = \mathbf{J} + \dfrac{\partial \mathbf{D}}{\partial t} .
\end{array} \right.$$

Then (15.2) rewrite as

$$\mathbf{D} = \epsilon_0 \frac{b}{a} \mathbf{E} , \qquad \mathbf{H} = \frac{1}{\mu_0} \frac{a}{b} \mathbf{B} ,$$

and they may be compared with

$$\mathbf{D} = \epsilon \epsilon_0 \mathbf{E} , \qquad \mathbf{H} = \frac{1}{\mu_0 \mu} \mathbf{B} ;$$

from which it follows

$$\epsilon = \frac{b}{a} , \qquad \mu = \frac{b}{a} .$$

The corresponding metric tensor (15.1) is

$$g_{\alpha\beta}(x) = b^2 \begin{pmatrix} k^2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} , \qquad k = \frac{1}{\sqrt{\epsilon\mu}} = \frac{a}{b} . \tag{15.3}$$

For simplicity, we further assume that

$$b = 1 , \qquad \epsilon = \mu = \frac{1}{a} = k, \qquad g^{\alpha\beta} = \begin{pmatrix} k^2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} . \tag{15.4}$$

16 Conclusion

In connection with these two theoretical alternative schemes, an essential issue must be stressed: it seems reasonable to perform the Poincaré-Einstein clock synchronization in uniform media with the help of real light signals influenced by the medium, which leads us to modified Lorentz symmetry.

In this context, it makes sense to specify a generally covariant Dirac equation in space-time determined by the metrics (15.3)–(15.4), assuming that in such a way we obtain the description of the Dirac particle in non-vacuum case, but in the uniform medium.

Starting with the general covariant form of the Dirac equation

$$\left\{ i\,\gamma^{a}\;\left[e^{\alpha}_{(a)}\,\frac{\partial}{\partial x^{\alpha}} + \frac{1}{2}\,\left(\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^{\alpha}}\sqrt{-g}\;e^{\alpha}_{(a)} \right) \right] \;-\; \frac{mc}{\hbar} \right\}\;\Psi(x) = 0\;. \tag{16.1}$$

with the tetrad

$$e^{\beta}_{(a)} = \begin{pmatrix} k^{-1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\;;$$

we readily get

$$\left(i\,\gamma^{0}\;\frac{1}{kc}\frac{\partial}{\partial t} + i\,\gamma^{j}\;\frac{\partial}{\partial x^{j}} -\;\frac{mc}{\hbar} \right)\;\Psi(x) = 0\;; \tag{16.2}$$

this means that eq. (16.2) can be viewed as a Dirac equation in the uniform medium where the real light velocity is kc.

Similarly, the generally covariant Klein-Fock-Gordon equation

$$\left(\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^{\alpha}}\sqrt{-g}g^{\alpha\beta}(x)\frac{\partial}{\partial x^{\beta}} + \frac{m^{2}c^{2}}{\hbar^{2}} \right)\,\Phi = 0 \tag{16.3}$$

related with the metrics (15.3)–(15.4) has the form

$$\left(\frac{1}{k^{2}c^{2}}\frac{\partial^{2}}{\partial t^{2}} - \nabla^{2} + \frac{m^{2}c^{2}}{\hbar^{2}} \right)\,\Phi = 0. \tag{16.4}$$

We can consider this equation as describing the scalar particle in the uniform medium as well.

This approach to scalar and spinor fields, equally as the electromagnetic field, is consistent with the assumption on the Poincaré-Einstein clock synchronization in uniform media with the help of real light signals influenced by the medium, which leads us to modified Lorentz symmetry.

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ЭЛЕКТРОМАГНИТНЫЕ УРАВНЕНИЯ МАКСВЕЛЛА В ОДНОРОДНОЙ СРЕДЕ. АЛЬТЕРНАТИВА ПОДХОДУ МИНКОВСКОГО В СПЕЦИАЛЬНОЙ ТЕОРИИ ОТНОСИТЕЛЬНОСТИ.

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Обсуждаются две альтернативные друг другу возможности представления электромагнитных уравнений Максвелла в движущейся однородной среде. Обычно используемый подход Минковского основывается на применении двух электромагнитных тензоров; связь между ними изменяет свой вид после применения преобразований Лоренца и принимает форму уравнений связи Минковского, которые зависят явным образом от 4-скорости системы отсчета. В этом подходе волновое уравнение для электромагнитного 4-потенциала содержит в себе 4-скорость системы отсчета. Следовательно, электродинамики Минковского подразумевает фактически абсолютный характер механического движения. Альтернативный формализм (предложенный Розеном и др.) может быть построен в новых переменных, при этом уравнения Максвелла записываются с помощью одного электромагнитного тензора. Эта форма уравнений Максвелла обладает симметрией относительно модифицированных преобразований Лоренца, в которых везде вместо скорости света в вакууме c используется скорость света в среде $c' < c$. В силу этой симметрии, формулировка теории Максвелла в среде может рассматриваться как инвариантная относительного механического движения системы отсчет, при этом преобразование скорости описывается модифицированными формулами Лоренца. Переход в уравнениях Максвелла к 4-потенциалу приводит к простому волновому уравнению , которое не содержит дополнительного параметра 4-скорости, т.е. эта форма электродинамики сохраняет относительную природу скорости механического движения. Также это уравнение описывает волны, распространяющиеся в пространстве со скоростью света kc, и эта скорость инвариантна относительно модифицированных преобразований Лоренца. В связи с существование этих двух теоретических альтернативных схем, может быть сформулировано существенное физическое положение: представляется разумным выполнять синхронизацию часов в однородной среде согласно Пуанкаре-Эйнштейну с помощью реальных световых сигналов в среде, что ведет к модифицированной симметрии Лоренца. Похожий подход развивается и для частицы со спином $1/2$, подчиняющейся уравнению Дирака в однородной среде.

Ключевые слова: электромагнитная теория, однородная среда, подход Минковского, модифицированная симметрия Лоренца.