

ON POSSIBLE EFFECTS OF THE SPINOR STRUCTURES IN QUANTUM PHYSICS

Elena Ovsiyuk¹, Olga Veko¹, Alexandru Oană², Mircea Neagu²,
Vladimir Balan³, Victor Red'kov⁴

¹ *Mozyr State Pedagogical University, Mozyr, Belarus*

² *University Transilvania of Braşov, Braşov, Romania*

³ *University Politehnica of Bucharest, Bucharest, Romania*

⁴ *B.I. Stepanov Institute of Physics, NAS of Belarus*

v.redkov@dragon.bas-net.by

The paper discusses the following topics: the spinor structure of space models; the relation between the Dirac–Schwinger quantization rule and the superposition principle in quantum mechanics; the manifestation of spinor space structure in classifying the solutions of the Dirac equation and for the matrix elements which are related to physical quantities; spinors in polarization optics; the Jones formalism for completely and partly polarized light; General Relativity and Riemannian space-time models with spinor structure and tetrad (vierbein) formalism.

Key Words: spinors, tetrads, Dirac–Schwinger quantization, Dirac equation, superposition principle, polarized light, Majorana 4-spinors, Lorentz group, relativistic fermion parity.

General introduction

In the literature, there exist three different terminological approaches, whose intrinsic essence is similar: a space-time spinor structure (Penrose, Rindler et al.); the Hopf and the Kustaanheimo-Stiefel bundles.

In Hopf's technique, one uses only complex 2-spinors (ξ) and their conjugates (ξ^*), instead of real-valued 4-vector (tensor) objects. In the Kustaanheimo-Stiefel approach, there are used four real-valued coordinates – the real and imaginary parts of 2-spinor components.

The formalism developed in the present work exploits as well the possibilities given by spinors to construct 3-vectors; however, the emphasis is put on doubling the set of spatial points, so that we get an extended space model. In such a space, instead of the 2π -rotation, there is considered the 4π -rotation - which transfers the space into itself. The procedure of doubling the manifold is achieved here much easier, while using curvilinear coordinates.

Within the framework of applications of spinor theory to Relativistic and Non-relativistic Physics, Quantum Mechanics and Polarization Optics, we discuss several actual issues, as:

- the concept of spinor structure in space-time models;
- exact linear representations for spinor coverings of the full Lorentz group;
- internal space-time parity of a relativistic fermion;
- Euclidean 3-spaces with opposite P -properties and two kinds of Cartan spatial spinors;
- parametrization of Cartan's spatial spinors by curvilinear coordinates, non-relativistic physics;
- the role of spinor space structure in classification of solutions of the Klein-Fock-Gordon equation and the influence on the matrix elements related to physical quantities.

1 Spin covering for the full Lorentz group $L_{+-}^{\uparrow\downarrow}$ and the concept of relativistic fermion parity

To treat the problem of fermion parity, we will use 4-spinors instead of 2-spinors. Additional motivation for this approach is that among 4-spinors there exist real-valued ones – the so-called Majorana 4-spinors; moreover, in this way we will be able to describe discrete symmetries by linear transformations¹.

The obtained results will provide the grounds for a new discussion of the old fermion parity problem of investigating possible linear single-valued representations of spinor coverings of the extended Lorentz group. It is shown that in the frame of this theory, P -parity and T -parity for a fermion do not exist as separate concepts; instead of these, only some unified concept of (PT) -parity can be described in a group-theoretical language.

We attach to the proper orthochronous Lorentz matrices

$$\begin{aligned} L_a^b(k, k^*) &= \bar{\delta}_c^b(-\delta_c^b k^n k_n^* + k_c k^{b*} + k_c^* k^b + i\epsilon_c^{bmn} k_m k_n^*), \\ L(k, k^*) &= L(-k, -k^*) \end{aligned} \tag{1.1}$$

two linear operations

$$P : L_a^{(P)b} = +\bar{\delta}_a^b ; \quad +T : L_a^{(T)b} = -\bar{\delta}_a^b,$$

where $\bar{\delta}_a^b = \text{diag}(+1, -1, -1, -1)$, of which one readily produces the full Lorentz group $L_{+-}^{\uparrow\downarrow}$. The commutation rules between $L_a^b(k, k^*)$ and the discrete elements P, T are

$$\bar{\delta}_a^b L_b^c(k, k^*) = L_a^b(\bar{k}^*, \bar{k}) \bar{\delta}_b^c. \tag{1.2}$$

The group $L_{+-}^{\uparrow\downarrow}$ has four types of vector representations:

$$T_a^b(L) = f(L) L_a^b, \quad L \in L_{+-}^{\uparrow\downarrow}. \tag{1.3}$$

namely

$$\begin{aligned} f_1(L) &= 1, & f_2(L) &= \det(L), \\ f_3(L) &= \text{sgn}(L_0^0), & f_4(L) &= \det(L) \text{sgn}(L_0^0). \end{aligned} \tag{1.4}$$

which have the explicit form

$$\begin{aligned} 1 : & \begin{aligned} T_1(L) &= L \\ T_1(P) &= +P \\ T_1(T) &= +T \end{aligned} & 2 : & \begin{aligned} T_2(L) &= L \\ T_2(P) &= -P \\ T_2(T) &= -T \end{aligned} \\ 3 : & \begin{aligned} T_3(L) &= L \\ T_3(P) &= +P \\ T_3(T) &= -T \end{aligned} & 4 : & \begin{aligned} T_4(L) &= L \\ T_4(P) &= -P \\ T_4(T) &= +T. \end{aligned} \end{aligned} \tag{1.5}$$

It should be emphasized that the above-described extension of the group $L_a^b(k, k^*)$ by adding the two discrete operations P and T is not an extension of the spinor group $SL(2, \mathbb{C})$: actually this is just an expansion of the orthogonal group L_+^{\uparrow} . From the spinor point of view, the operations P and T are transformations which act on the space of 2-rank spinors, and *not* on the space of 1-rank spinors. Evidently, a more comprehensive study of P, T -symmetry can be done in the framework of first-rank spinors, when one extends the covering group $SL(2, \mathbb{C})$ by adding spinor discrete operations.

¹Mainly, we will consider only the problem of accurate description of the single-valued representations of four different spinor groups, each of them covering the full Lorentz group $L_{+-}^{\uparrow\downarrow}$, including P and T -reflections.

Now we can start solving this task. A covering group for the total Lorentz group can be constructed by adding two specific 4×4 -matrices to the known set of 4-spinor transformations of the group $SL(2, \mathbb{C})$,

$$S(k, \bar{k}^*) = \begin{pmatrix} B(k) & 0 \\ 0 & B(\bar{k}^*) \end{pmatrix} \in \widetilde{SL}(2, \mathbb{C}). \tag{1.6}$$

Those two new matrices are to be taken from the following

$$M = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad M' = iM, \quad N = \begin{pmatrix} 0 & -iI \\ +iI & 0 \end{pmatrix}, \quad 'N = iN. \tag{1.7}$$

Having added any two elements of the four ones, we provide the full extension of the group $\widetilde{SL}(2, \mathbb{C})$, by means of two new operations only. Also, we note that since the group $L(2, \mathbb{C})$ contains $-I$, the extension of the group by any two elements of $\{-M, -M', -N, -'N\}$, leads to the same result. However, if one takes any other phase factor, different from $+1, -1, +i, -i$ for $M, M', N, 'N$, then this will result in substantially new extended groups.

The multiplication table for these four discrete elements is

	M	M'	N	$'N$	
M	$\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$	$\begin{pmatrix} iI & 0 \\ 0 & iI \end{pmatrix}$	$\begin{pmatrix} +iI & 0 \\ 0 & -iI \end{pmatrix}$	$\begin{pmatrix} -I & 0 \\ 0 & +I \end{pmatrix}$	
M'	$\begin{pmatrix} +iI & 0 \\ 0 & +iI \end{pmatrix}$	$\begin{pmatrix} -I & 0 \\ 0 & -I \end{pmatrix}$	$\begin{pmatrix} -I & 0 \\ 0 & +I \end{pmatrix}$	$\begin{pmatrix} -iI & 0 \\ 0 & +iI \end{pmatrix}$	(1.8)
N	$\begin{pmatrix} -iI & 0 \\ 0 & +iI \end{pmatrix}$	$\begin{pmatrix} +I & 0 \\ 0 & -I \end{pmatrix}$	$\begin{pmatrix} +I & 0 \\ 0 & +I \end{pmatrix}$	$\begin{pmatrix} +iI & 0 \\ 0 & +iI \end{pmatrix}$	
$'N$	$\begin{pmatrix} +I & 0 \\ 0 & -I \end{pmatrix}$	$\begin{pmatrix} +iI & 0 \\ 0 & -iI \end{pmatrix}$	$\begin{pmatrix} +iI & 0 \\ 0 & +iI \end{pmatrix}$	$\begin{pmatrix} -I & 0 \\ 0 & -I \end{pmatrix}$	

Hence we obtain six covering groups,

$$\begin{aligned} G_M &= \{ S(k, \bar{k}^*) \uplus M \uplus M' \}, & G_N &= \{ S(k, \bar{k}^*) \uplus N \uplus 'N \}, \\ G' &= \{ S(k, \bar{k}^*) \uplus M' \uplus N \}, & 'G &= \{ S(k, \bar{k}^*) \uplus 'N \uplus M \}, \\ G &= \{ S(k, \bar{k}^*) \uplus M \uplus N \}, & 'G' &= \{ S(k, \bar{k}^*) \uplus M' \uplus 'N \}, \end{aligned} \tag{1.9}$$

with the corresponding multiplication tables

$$\begin{aligned} G_M : & \quad M^2 = +I, \quad M'^2 = -I, & \quad MM' &= (M')M; \\ G_N : & \quad N^2 = +I, \quad 'N^2 = -I, & \quad N('N) &= ('N)N; \\ G' : & \quad M'^2 = -I, \quad N^2 = +I, & \quad (M')N &= -N(M'); \\ 'G : & \quad ('N)^2 = -I, \quad M^2 = +I, & \quad ('N)M &= -M('N); \\ G : & \quad M^2 = +I, \quad N^2 = +I, & \quad MN &= -NM; \\ 'G' : & \quad (M')^2 = -I, \quad N'^2 = -I, & \quad (M')('N) &= -('N)(M'), \end{aligned} \tag{1.10}$$

and

$$F S(k, \bar{k}^*) = S(\bar{k}^*, k) F, \quad F \in \{M, M', N, 'N\}. \tag{1.11}$$

One can notice that the multiplication laws for the groups G_M and G_N happen to coincide; the same happens for G' and $'G$. This implies that the groups G_M and G_N (and respectively

G' and $'G$) represent the same abstract group. Indeed, it is readily verified that G_M and G_N (and, also, G' and $'G$), can be transformed into each other by a similarity transformation:

$$\begin{aligned} G_N = A G_M A^{-1} : \quad & A S(k, \bar{k}^*) = S(\bar{k}^*, k) A , \\ & A M A^{-1} = +N , \quad A M' A^{-1} = +'N , \\ & A = \text{const} \cdot \begin{pmatrix} -iI & 0 \\ 0 & +I \end{pmatrix} ; \end{aligned} \tag{1.12}$$

$$\begin{aligned} 'G = A G' A^{-1} : \quad & A S(k, \bar{k}^*) = S(\bar{k}^*, k) A , \\ & A M' A^{-1} = +'N , \quad A N A^{-1} = -M , \\ & A = \text{const} \cdot \begin{pmatrix} -iI & 0 \\ 0 & +I \end{pmatrix} . \end{aligned} \tag{1.13}$$

In other words, we define here only four different covering groups. Since in literature all the six variants are discussed, we shall accordingly trace all of them.

1.2. Representations of the extended spinor groups

We shall construct now the exact linear representations of the groups $G_M, G_N, G', 'G, G, 'G'$. It suffices to consider in detail only one group; for convenience, let this be G_M . Its multiplication table is

$$\begin{aligned} M^2 = -I , \quad M'^2 = -I , \quad M M' = M M , \\ F S(k, \bar{k}^*) = S(\bar{k}^*, k) F , \quad (F = M , M') , \\ (k_1, \bar{k}_1^*)(k_2, \bar{k}_2^*) = (< k_1, k_2 > , < \bar{k}_1^*, \bar{k}_2^* >) . \end{aligned} \tag{1.14}$$

where the symbol $< , >$ stands for the known multiplication rule in the group $SL(2, \mathbb{C})$:

$$< k_1, k_2 > = (k_1^0 k_2^0 + \vec{k}_1 \vec{k}_2; \vec{k}_1 k_2^0 + k_2^0 \vec{k}_1 + i[\vec{k}_1 \vec{k}_2]) . \tag{1.15}$$

Let us look for solution of the problem $g \rightarrow T(g)$ in the form

$$T(g) = f(g) g , \quad g \in G_M , \quad f(g_1) \cdot f(g_2) = f(g_1 \cdot g_2) \tag{1.16}$$

where $f(g)$ is a numerical function on the group G_M . Substitution (1.16) into (1.14) yields

$$\begin{aligned} [f(M)]^2 = f(I) , \quad [f(M')]^2 = f(-I) , \quad f(S(k, \bar{k}^*)) = f(S(\bar{k}^*, k)) , \\ f(S(k_1, \bar{k}_1^*)) f(S(k_2, \bar{k}_2^*)) = f(S(< k_1, k_2 > , < \bar{k}_1^*, \bar{k}_2^* >)) . \end{aligned} \tag{1.17}$$

There exist four different such functions f_i , described by:

$$\begin{array}{ccccc} G_M & f_1(g) = & f_2(g) = & f_3(g) = & f_4(g) = \\ S(k, \bar{k}^*) & +1 & +1 & +1 & +1 \\ M & +1 & -1 & +1 & -1 \\ M' & +1 & -1 & -1 & +1, \end{array} \tag{1.18}$$

which provide four representations $T_i(g)$ of the group G_M .

In the same manner, one can construct the analogous representation $T_i(g)$ of the remaining

five groups. All these are described by the following table

	g	$T_1(g)$	$T_2(g)$	$T_3(g)$	$T_4(g)$	
	$S(k, \bar{k}^*)$	$S(k, \bar{k}^*)$	$S(k, \bar{k}^*)$	$S(k, \bar{k}^*)$	$S(k, \bar{k}^*)$	
G_M	M	$+M$	$-M$	$+M$	$-M$	
	M'	$+M'$	$-M'$	$-M'$	$+M'$	
G_N	N	$+N$	$-N$	$+N$	$-N$	
	$'N$	$+'N$	$- 'N$	$- 'N$	$+'N$	
G'	M'	$+M'$	$-M'$	$+M'$	$-M'$	(1.19)
	N	$+N$	$-N$	$-N$	$+N$	
$'G$	$'N$	$+'N$	$- 'N$	$+'N$	$- 'N$	
	M	$+M$	$-M$	$-M$	$+M$	
G	M	$+M$	$-M$	$+M$	$-M$	
	N	$+N$	$-N$	$+N$	$-N$	
$'G'$	M'	$+M'$	$-M'$	$+M'$	$-M'$	
	$'N$	$+'N$	$- 'N$	$- 'N$	$+'N$	

For each of these groups, one can ask whether the four representations $T_i(g)$ are equivalent, or not. With the help of the relations

$$\begin{aligned}
 F = const \begin{pmatrix} -I & 0 \\ 0 & +I \end{pmatrix}, & \quad F S(k, \bar{k}^*) F^{-1} = S(k, \bar{k}^*), \\
 F M F^{-1} = -M, & \quad F M' F^{-1} = -M' F, \\
 N F^{-1} = -N, & \quad F 'N F^{-1} = - 'N,
 \end{aligned}
 \tag{1.20}$$

it easily follows that the type $T_2(g)$ is equivalent to the type $T_1(g)$, as well, $T_4(g)$ is equivalent to $T_3(g)$:

$$T_2(g) = F T_1(g) F^{-1}, \quad T_4(g) = F T_3(g) F^{-1}.
 \tag{1.21}$$

Summarizing, we have got to the following: for each of the six groups, only two non-equivalent representations $g \rightarrow T(g) = f(g) g$ are possible:

$$T_1(g) \sim T_2(g), \quad T_3(g) \sim T_4(g).
 \tag{1.22}$$

Evidently, this result does not depend on the explicit realization of the discrete spinor transformations.

The above study of the exact linear representations of the extended spinor groups leads to a new concept of a space-time intrinsic parity of a fermion. In group-theoretical terms P -parity and T -parity do not have any sense, instead only their joint characteristic, that might be called (PT) -parity, can be defined in the group-theoretic framework.

1.3. Representations of the coverings for partly extended groups L_{+-}^\uparrow and $L_+^{\uparrow\downarrow}$

Now we are going to consider the problem of linear representations of the spinor groups that cover the partly extended Lorentz groups L_{+-}^\uparrow and $L_+^{\uparrow\downarrow}$ (improper orthochronous and proper non-orthochronous, respectively). Such groups can be constructed by adding any matrix from $\{M, M', N, 'N\}$.

The case of the orthogonal group L_{+-}^\uparrow leads to

$$T_1 = T_3 ; L \implies L = (\text{sgn } L_0^0) L , \tag{1.23}$$

$$T_2 = T_4 ; L \implies L = (\det L)L = (\det L)(\text{sgn } L_0^0) L , \tag{1.24}$$

and the case of the group $L_+^{\uparrow\downarrow}$ looks as

$$T_1 = T_4 ; L \implies L = (\det L)(\text{sgn } L_0^0) L , \tag{1.25}$$

$$T_2 = T_3 ; L \implies L = (\det L) L = (\text{sgn } L_0^0) L . \tag{1.26}$$

With the use of one additional discrete operation, one can determine four extended spinor groups:

$$\widetilde{SL}(2, \mathbb{C})_M = \{ S(k, \bar{k}^*) \oplus M \} \text{ and so on .} \tag{1.27}$$

We conclude that the extended groups $\widetilde{SL}(2, \mathbb{C})_M$, $\widetilde{SL}(2, \mathbb{C})_N$ turn out to be isomorphic. Analogously, $\widetilde{SL}(2, \mathbb{C})_{M'}$ is isomorphic to $\widetilde{SL}(2, \mathbb{C})_{N'}$. Each of them covers both $L_+^{\uparrow\downarrow}$ and L_{+-}^\uparrow :

$$\widetilde{SL}(2, \mathbb{C})_M \sim \widetilde{SL}(2, \mathbb{C})_N , \quad \widetilde{SL}(2, \mathbb{C})_{M'} \sim \widetilde{SL}(2, \mathbb{C})_{N'} . \tag{1.28}$$

Now, we shall list the simplest representations of these groups. The obtained result is as follows: all the representations $T_i(g)$ from above, while confining them to sub-groups $SL(2, \mathbb{C})_{M(N)}$ and $SL(2, \mathbb{C})_{M',(N')}$, lead to representations changing into each other by a similarity transformation. In other words, in fact there exists only one representation of these partly extended spinor groups. This may be understood as the impossibility to determine any group-theoretical parity concept (P or T) within the limits of partly extended spinor groups.

1.4. On reducing spinor groups to a real form

Till now we have considered all the spinor groups $G_M \sim G_N$, $G' \sim G$, G , G' as possible group covering candidates to for the full Lorentz group $L_{+-}^{\uparrow\downarrow}$. It is desirable to formulate some extra arguments in order to choose only *one* spinor group as a natural (physical) covering.

Note that in the bispinor space a special basis can be found using the bispinor wave function

$$\Phi_M(x) = \varphi(x) + i\xi(x) , \tag{1.29}$$

which transforms under the action of the group $SL(2, \mathbb{C})$ by means of real (4×4) -matrices. Therefore, the real 4-spinors $\varphi(x)$ and $\xi(x)$, constituents of the complex-valued $\Phi_M(x)$, transform as independent irreducible representations. In physical context of real Majorana fermions, this reads as a group-theoretical permission to exist. But these arguments have been based only on continuous $SL(2, \mathbb{C})$ -transformations, while the idea is to extend them on discrete operations too. So we must find the answer to the question of which of the extended spinor groups can be reduced to a real form. With this goal in mind, we write down the bispinor matrix a the form that does not depend on the randomly chosen basis²:

$$S = \frac{1}{2}(k_0 + k_0^*) + \frac{1}{2}(k_0 - k_0^*)\gamma^5 + (k_1 + k_1^*)\sigma^{01} + (k_1 - k_1^*)i\sigma^{23} + (k_2 + k_2^*)\sigma^{02} + (k_2 - k_2^*)i\sigma^{31} + (k_3 + k_3^*)\sigma^{03} + (k_3 - k_3^*)i\sigma^{13} . \tag{1.30}$$

Any Majorana basis satisfies the relations

$$(\gamma_M^a)^* = -\gamma_M^a , \quad (\gamma_M^5)^* = -\gamma_M^5 , \quad (\sigma_M^{ab})^* = \sigma_M^{ab} \implies S^* = S . \tag{1.31}$$

²We employed above the Weyl basis.

It remains to write down all the used discrete operations $M, M'N, 'N$ in terms of Dirac matrices:

$$M = +\gamma^0, \quad M' = +i\gamma^0, \quad N = +i\gamma^5\gamma^0, \quad 'N = -\gamma^5\gamma^0. \quad (1.32)$$

In Majorana frames, the discrete operations obey the following properties

$$S^* = S, \quad M^* = -M, \quad (M')^* = +M', \quad N^* = -N, \quad ('N)^* = +'N. \quad (1.33)$$

Thus, the six spinor groups behave under complex conjugation as indicated below

$$\begin{array}{cccccc} G_M & G_N & G' & 'G & G & 'G' \\ S^* = S & S^* = S & S^* = S & S^* = S & S^* = S & S^* = S \\ M^* = -M & N^* = -N & M'^* = +M' & 'N^* = +'N & M^* = -M & M'^* = +M' \\ (M')^* = M' & 'N^* = +'N & N^* = -N & M^* = -M & N^* = -N & 'M^* = 'M \end{array} \quad (1.34)$$

Only the group $'G'$ can be reduced to a real-valued form, and only this group allows real-valued spinor representations, namely the Majorana fermions³.

Conclusion

The problem of fermion parity is considered on the base of investigating possible single-valued representations of spinor coverings of the extended Lorentz group. It is shown that in the frame of this theory, there do not exist – as separate concepts – P -parity and T -parity for a fermion; instead of this, only some unified concept of (PT) -parity can be determined in group-theoretical terms. Apparently, physics with spinor group significantly differs from the one based on the orthogonal group $L_{+-}^{\uparrow\downarrow}$, and only experiment can decide on this problem. It is needless to say that this task cannot be solved without a thorough theoretical analysis of possible experimental verifications, in both orthogonal and spinor approaches.

2 Geometry of 3-spaces with spinor structure

Our special approach to examine the spinor structure of 3-space is based on the concept of spatial spinor, defined through taking the "square root" of a real-valued 3-vector. Two sorts of spatial spinors, according to the P -orientation of an initial 3-space, are introduced: proper-vector or pseudo-vector ones. These spinors, η and ξ , turn out to be different functions of Cartesian coordinates. To have a spinor space model, one has to use a doubling vector space $\{ (x_1, x_2, x_3) \otimes (x_1, x_2, x_3)' \}$. The information which is reachable here in the first place concerns non-relativistic physics in the frames of ideas on spinor space structure.

Spinor functions are in one-to-one correspondence with coordinates $x_i \oplus x'_i$, with the whole axis

$$(0, 0, x_3) \oplus (0, 0, x_3)'$$

removed; they exhibit an exponential discontinuity. Due to this reason, we shall consider the properties of spinor fields $\xi(x_i \oplus x'_i)$ and $\eta(x_i \oplus x'_i)$ in terms of continuity with respect to geometrical directions in the neighborhood of every point. This points out the possible fruitful geometrization within the Finslerian framework.

We shall further examine two sorts of spatial spinors, with the use of: cylindrical parabolic, spherical and parabolic coordinates. Transition from vector to spinor models is achieved by doubling the parameterizing domain $G(y_1, y_2, y_3) \implies \tilde{G}(y_1, y_2, y_3)$ with new identification rules on the boundaries. The differential equations satisfied by spatial spinor fields have been explicitly constructed. The use of curvilinear coordinates makes it easier to extend the formalism to curved (pseudo-Riemannian) models.

³This variant coincides with the known in the literature Racah group.

2.1. Two sorts of spatial spinors

We will start with the well-known Cartan's classification of 2-spinors with respect to the spinor P -reflection:

$$\widetilde{SU}(2) = \left\{ g \in SU(2) \oplus J = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} \mid \det g = +1, \det J = -1 \right\}, \quad (2.1)$$

which provides 2-component spinors of two sorts T_A :

$$T_1 : T_1(g) = g, T_1(J) = +J, \quad T_2 : T_2(g) = g, T_2(J) = -J. \quad (2.2)$$

There exist two ways to construct 3-vectors (complex-valued, in general) in terms of these 2-spinors:

$$1. (\xi \otimes \xi^*) = a + a_j \sigma^j, \quad a = \sqrt{a_j a_j}, \text{ pseudo-vector}; \quad (2.3)$$

$$2. (\eta \otimes \eta) = (c_j + i b_j) \sigma^j, \text{ vector}. \quad (2.4)$$

According to the way of taking the square root of the three real numbers – components of a 3-vector (x_i) , one obtains two different spatial spinors

$$\xi \iff a_j, \quad \eta \iff c_j \text{ or } (b_j). \quad (2.5)$$

2.2. The pseudo-vector space Π_3 and the spatial spinor ξ

This spinor model is based on the mapping

$$\Pi_3 \implies (a_1, a_2, a_3) \oplus (a_1, a_2, a_3)' \\ \xi = \begin{pmatrix} \sqrt{a + a_3} e^{-i\gamma/2} \\ \sqrt{a - a_3} e^{+i\gamma/2} \end{pmatrix}, \quad e^{i\gamma} = \frac{a_1 + ia_2}{\sqrt{a_1^2 + a_2^2}}. \quad (2.6)$$

It should be noted that in describing Π_0^+ and Π_0^- there arise peculiarities: at the whole axis a_3 , the relations contain the ambiguity $(0 + i0)/0$ (and the expressions for ξ will contain a mute angle variable $\Gamma : \gamma \rightarrow \Gamma$)

$$\Pi_0^+ : \quad \xi_0^+ = \begin{pmatrix} \sqrt{+2a_3} e^{-i\Gamma/2} \\ 0 \end{pmatrix}, \quad \Pi_0^- : \quad \xi_0^- = \begin{pmatrix} 0 \\ \sqrt{-2a_3} e^{+i\Gamma/2} \end{pmatrix}, \\ e^{i\Gamma} = \lim_{a_1, a_2 \rightarrow 0} \frac{a_1 + ia_2}{\sqrt{a_1^2 + a_2^2}}, \quad a_3 = 0, \quad \xi = \begin{pmatrix} \sqrt{a_1^2 + a_2^2} e^{-i\gamma/2} \\ \sqrt{a_1^2 + a_2^2} e^{+i\gamma/2} \end{pmatrix}. \quad (2.7)$$

2.3. The proper vector space E_3 and the spinor \vec{E}_3

This type of spatial spinor is based on the map

$$(\eta \otimes \eta) = (c_j + i b_j) \sigma^j. \quad (2.8)$$

The vector \vec{b} covers the upper half-space E_3^+ twice; the spinor η^+ is given by

$$\eta^+ = \begin{pmatrix} \sqrt{b - (b_1^2 + b_2^2)^{1/2}} e^{-i\gamma/2} \\ \sqrt{b - (b_1^2 + b_2^2)^{1/2}} e^{+i\gamma/2} \end{pmatrix}, \quad e^{i\gamma} = \frac{b_1 + ib_2}{\sqrt{b_1^2 + b_2^2}}. \quad (2.9)$$

The vector \vec{b} covers a down half-space E_3^- twice; the spinor η^- is

$$\eta^- = \begin{pmatrix} \sqrt{b - (b_1^2 + b_2^2)^{1/2}} \left[-\sqrt{\frac{b_1 + ib_2}{(b_1^2 + b_2^2)^{1/2}}} \right]^* \\ \sqrt{b + (b_1^2 + b_2^2)^{1/2}} \left[+\sqrt{\frac{b_1 + ib_2}{(b_1^2 + b_2^2)^{1/2}}} \right] \end{pmatrix}. \quad (2.10)$$

The spinor field η is continuous in the plane $b_3 = 0$:

$$\eta^{+\cap-} = \begin{pmatrix} 0 \\ \sqrt{2(b_1 + i b_2)} \end{pmatrix}. \quad (2.11)$$

2.4. The spatial spinor $\xi_{a_3}(a_1 + ia_2)$ and Cauchy-Riemann analyticity

It is natural to regard the components of spinor $\xi = \xi(a_j)$ as complex-valued functions of $z = a_1 + ia_2$ and of a real-valued function a_3 :

$$\xi^1 = U^1 + iV^1, \quad \xi^2 = U^2 + iV^2. \quad (2.12)$$

We obtain the modified Cauchy-Riemann relations

$$\begin{aligned} \frac{\partial U^1}{\partial a_1} - \frac{\partial V^1}{\partial a_2} &= \frac{1}{2}(a_1 \cos \frac{\gamma}{2} + a_2 \sin \frac{\gamma}{2}) \left[\frac{1}{a\sqrt{a+a_3}} + \frac{\sqrt{a+a_3}}{\rho^2} \right], \\ \frac{\partial U^1}{\partial a_2} + \frac{\partial V^1}{\partial a_1} &= \frac{1}{2}(a_2 \cos \frac{\gamma}{2} - a_1 \sin \frac{\gamma}{2}) \left[\frac{1}{a\sqrt{a+a_3}} + \frac{\sqrt{a+a_3}}{\rho^2} \right], \\ \frac{\partial U^2}{\partial a_1} - \frac{\partial V^2}{\partial a_2} &= \frac{1}{2}(a_1 \cos \frac{\gamma}{2} - a_2 \sin \frac{\gamma}{2}) \left[\frac{1}{a\sqrt{a-a_3}} - \frac{\sqrt{a-a_3}}{\rho^2} \right], \\ \frac{\partial U^2}{\partial a_2} + \frac{\partial V^2}{\partial a_1} &= \frac{1}{2}(a_2 \cos \frac{\gamma}{2} + a_1 \sin \frac{\gamma}{2}) \left[\frac{1}{a\sqrt{a-a_3}} - \frac{\sqrt{a-a_3}}{\rho^2} \right]. \end{aligned} \quad (2.13)$$

For $\rho \rightarrow \infty$, the Cauchy-Riemann conditions still hold true.

A special note should be given to the behavior of the spinor field ξ^i along the half-plane $\{a_1 \geq 0, a_2 = 0\}^{a_3}$. Here the spinor ξ is not a single-valued function of spatial points of the vector space Pi_3 because its values depend on the direction from which one approaches those points.

2.5. Calculating $\nabla \xi$ and $\nabla_{\vec{n}} \xi$. The differential equation

The spinor exhibit continuity properties. In order to point them out, let us calculate first the 2-gradient along an arbitrary direction

$$\nabla \xi = \left(\frac{\partial}{\partial a_1} \xi, \frac{\partial}{\partial a_2} \xi \right), \quad \nabla_{\vec{n}} \xi = (\vec{n} \nabla \xi) \quad (2.14)$$

in the neighborhood of an arbitrary point⁴:

$$\begin{aligned} \nabla_{\vec{n}} \xi^1 &= \frac{1}{2} \left[\frac{(\vec{n} \vec{a})}{a(a+a_3)} + i \frac{\vec{n} \times \vec{a}}{\rho^2} \right] \xi^1, \\ \nabla_{\vec{n}} \xi^2 &= \frac{1}{2} \left[\frac{(\vec{n} \vec{a})}{a(a-a_3)} - i \frac{\vec{n} \times \vec{a}}{\rho^2} \right] \xi^2. \end{aligned} \quad (2.15)$$

This can be considered as a basic equation that prescribes the explicit form of the spinor $\xi(\vec{a})$. This understanding seems to be interesting, since it allows the existing of a mathematical potential.

⁴we use the notation $\vec{n}\vec{a} = n_1a_1 + n_2a_2, \vec{n} \times \vec{a} = n_1a_2 - n_2a_1$.

2.6. Spinor η and its differential equation

Similar things can be done for other spatial spinors. Again, we derive the differential equations for a spatial spinor of that type

$$\begin{aligned} \nabla_{\vec{n}}\eta^1 &= \frac{1}{2\rho} \left[-\frac{1}{b}(\vec{n} \vec{b}) + \frac{i}{\rho}(\vec{n} \times \vec{b}) \right] \eta^1, \\ \nabla_{\vec{n}}\eta^2 &= \frac{1}{2\rho} \left[+\frac{1}{b}(\vec{n} \vec{b}) - \frac{i}{\rho}(\vec{n} \times \vec{b}) \right] \eta^2. \end{aligned} \tag{2.16}$$

2.7. Comparison of the models ξ and η

We shall further describe several qualitative distinctions between the spinor models ξ and η . The two models of spinor spaces relative to the P -orientation rely on the different mappings ξ and η defined over the same extended domain $\tilde{G}(y_i)$. The natural question is: how do these two maps relate? An answer can be found by comparing the derived formulas for ξ and η .

One answer emerges straightforward:

$$\eta = \frac{1}{\sqrt{2}}(\xi - i \sigma^2 \xi^*) \quad \text{or inverse} \quad \xi = \frac{1}{\sqrt{2}}(\eta - i \sigma^2 \eta^*). \tag{2.17}$$

An issue which needs special attention is the fact that complex conjugation enters these relations explicitly, fact which correlates to the change in orientation properties of the models.

We have seen that the description of differently P -oriented geometries in terms of spinor fields η and ξ has made hardly noticeable the distinction between these two geometries - much more apparent and intuitively appreciable as connected with different types of spatial geometry.

2.8. Spinors ξ and η in cylindrical parabolic coordinates

This coordinate system in the vector E_3 -space is defined by

$$\begin{aligned} x_1 &= \frac{y_1^2 - y_2^2}{2}, \quad x_2 = y_1 y_2, \quad x_3 = y_3, \\ y_2 &\in [0, +\infty), \quad y_1, y_3 \in (-\infty, +\infty). \end{aligned} \tag{2.18}$$

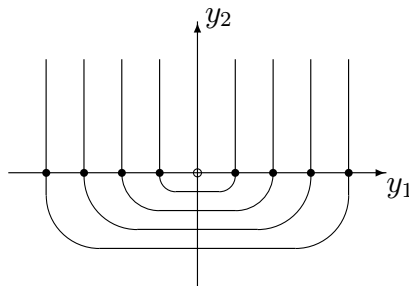


Fig. 1. Parabolic cylindrical coordinates

where the identified points on the boundary are connected by lines, and the domain $G(y_1, y_2)^{y_3}$ (at arbitrary y_3) ranging in the half-plane (y_1, y_2) covers the whole vector plane $(x_1, x_2)^{x_3}$.

The spinor ξ of the pseudo-vector Π_3 -model is given by

$$\xi = \left(\begin{array}{c} \sqrt{(y_3^2 + (y_1^2 + y_2^2)^2/4)^{1/2} + y_3} e^{-i\gamma/2} \\ \sqrt{(y_3^2 + (y_1^2 + y_2^2)^2/4)^{1/2} - y_3} e^{+i\gamma/2} \end{array} \right), \quad e^{i\gamma/2} = \frac{y_1 + iy_2}{\sqrt{y_1^2 + y_2^2}}, \tag{2.19}$$

the factor $e^{i\gamma/2}$ runs through the upper complex half-plane in the case of vector space. At the x_3 -axis, we have:

$$\xi_0^+ = \sqrt{+2y_3} \begin{pmatrix} e^{-i\Gamma/2} \\ 0 \end{pmatrix}, \quad \xi_0^- = \sqrt{-2y_3} \begin{pmatrix} 0 \\ e^{+i\Gamma/2} \end{pmatrix}, \quad (2.20)$$

For a proper vector model, the η -spinor looks as⁵

$$\eta^\sigma(y) = \begin{pmatrix} \sqrt{\sqrt{y_3^2 + (y_1^2 + y_2^2)^2/4} - \frac{y_1^2 + y_2^2}{2}} \sigma e^{-i\gamma/2} \\ \sqrt{\sqrt{y_3^2 + (y_1^2 + y_2^2)^2/4} + \frac{y_1^2 + y_2^2}{2}} e^{-i\gamma/2} \end{pmatrix}. \quad (2.21)$$

We construct the extended (spinor) models \tilde{E}_3 and $\tilde{\Pi}_3$ by doubling the range of the y_2 -variable:

$$y_2 \in [0, +\infty) \implies y_2 \in (-\infty, +\infty). \quad (2.22)$$

Then the above factor $e^{+i\gamma/2}$ will run through the full circle.

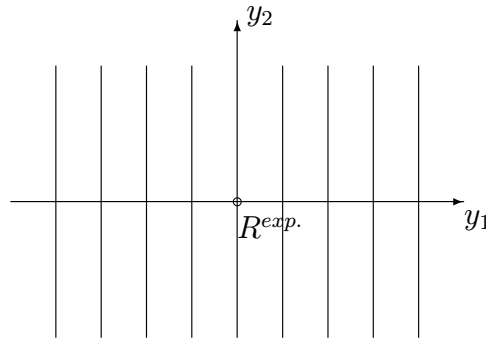


Fig. 2. Space with spinor structure

It is important to note the substantial change in the identification rules at the boundary set of $G(y_1, y_2, y_3)$: for the extended domain $\tilde{G}(y_1, y_2, y_3)$ one needs no special rules at all. Another issue needs to be emphasized: we have the same extended set $\tilde{G}(y_1, y_2, y_3)$ for both spinor models $\xi(y)$ and $\eta(y)$. This means that only the providing of the set with doubling dimension and the using of identification rules, do not determine in full the whole geometry of the spinor spaces. The specification of their P -orientation apparently requires additional information about this set. Evidently, P -orientation manifests itself in the explicitly different spinor functions $\xi(y)$ and $\eta(y)$. Moreover, a qualitative distinction between these spinor functions is revealed if one follows the orientation of a spinor (ξ_1, ξ_2) and (η_1, η_2) , while going from the x_3^+ - half-space to the x_3^- - half-space.

The differential equations for spatial spinors are

$$\begin{aligned} \nabla_{\vec{v}} \xi^1 &= \frac{\xi^1}{2} \left[\frac{\rho}{a(a + a_3)} (\vec{n} \vec{y}) + \frac{i}{\rho} (\vec{n} \times \vec{y}) \right], \\ \nabla_{\vec{n}} \xi^2 &= \frac{\xi^2}{2} \left[\frac{\rho}{a(a - a_3)} (\vec{n} \vec{y}) - \frac{i}{\rho} (\vec{n} \times \vec{y}) \right], \end{aligned} \quad (2.23)$$

and⁶

$$\nabla_{\vec{n}} \eta^1 = \frac{\eta^1}{2} \left[-\frac{\vec{n} \vec{y}}{b} + \frac{i}{\rho} (\vec{n} \times \vec{y}) \right], \quad \nabla_{\vec{n}} \eta^2 = \frac{\eta^2}{2} \left[\frac{\vec{n} \vec{y}}{b} - \frac{i}{\rho} (\vec{n} \times \vec{y}) \right], \quad (2.24)$$

These equations have no peculiarities over the complex plane $y_1 + iy_2$, excluding the origin $0 + i0$.

⁵The values + and - taken by the symbol σ correspond to the $x_3 > 0$ and $x_3 < 0$ half-spaces, respectively.

⁶We denote $(\vec{n} \vec{y}) = n_1 y_1 + n_2 y_2$, $\vec{n} \times \vec{y} = n_1 y_2 - n_2 y_1$.

2.9. The spinors ξ and η in parabolic coordinates

We shall further describe the spinor approach relative to the well-known parabolic coordinates

$$\begin{aligned} x_1 &= y_1 y_2 \cos y_3, \quad x_2 = y_1 y_2 \sin y_3, \\ x_3 &= \frac{y_1^2 - y_2^2}{2}, \quad y_1, y_2 \in [0, +\infty), \quad y_3 \in [0, 2\pi] \end{aligned} \tag{2.25}$$

The spatial spinor η of the proper vector model is given by

$$\eta^+(y) = \frac{1}{\sqrt{2}} \begin{pmatrix} (y_1 - y_2) e^{-iy_3/2} \\ (y_1 + y_2) e^{+iy_3/2} \end{pmatrix}, \quad \eta^-(y) = \frac{1}{\sqrt{2}} \begin{pmatrix} (y_2 - y_1) (-e^{-iy_3/2}) \\ (y_2 + y_1) e^{+iy_3/2} \end{pmatrix}. \tag{2.26}$$

As for the pseudo-vector model Π_3 , we have

$$\xi = \begin{pmatrix} y_1 e^{-iy_3/2} \\ y_2 e^{+iy_3/2} \end{pmatrix}, \quad \xi = \begin{pmatrix} N e^{-i\gamma/2} \\ M e^{+i\gamma/2} \end{pmatrix}, \quad y_1 = N, y_2 = M, y_3 = \gamma. \tag{2.27}$$

We double the above domain $G(y) \Rightarrow \tilde{G}(y) (y_3 \in [-2\pi, +2\pi])$,

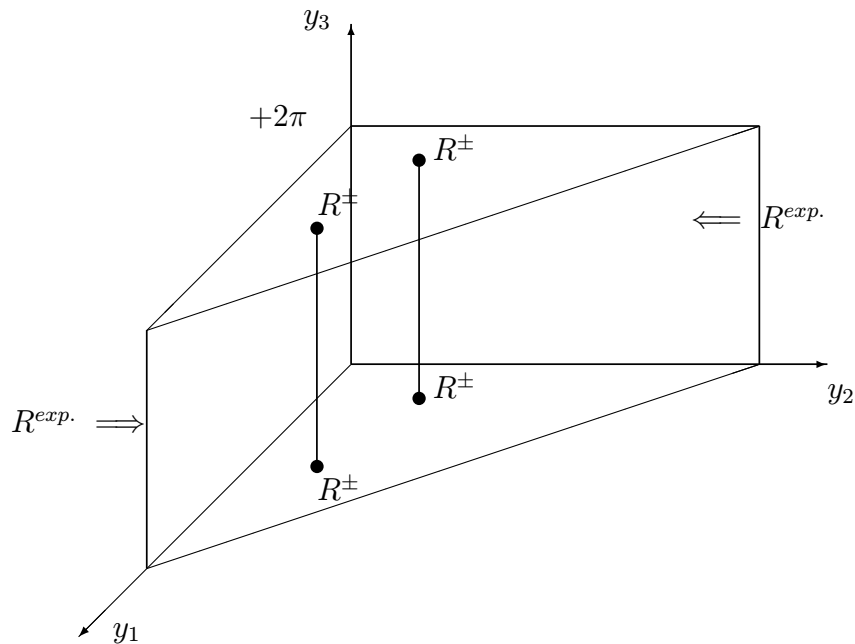


Fig. 3. Parabolic coordinates / vector space

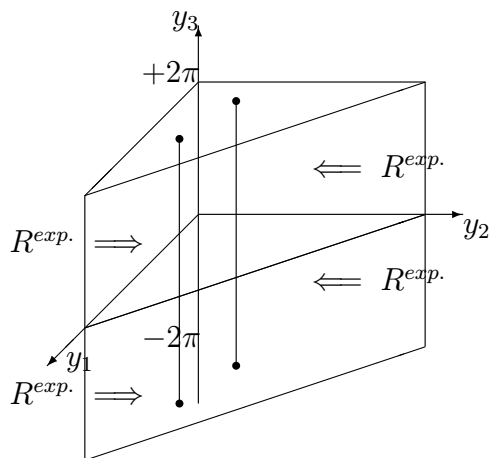


Fig. 4. Parabolic coordinates / spinor space

Instead of the domain $\tilde{G}(y)$ described below

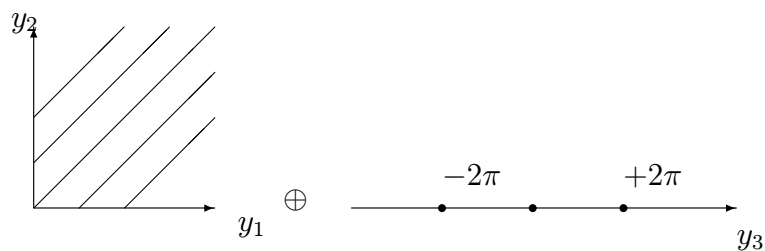


Fig. 5. Domain parameterizing the spinor space

one can use

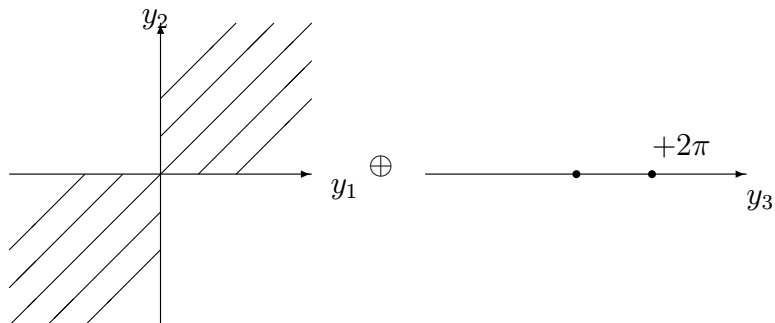


Fig. 6. Alternative domain to parameterize the spinor space

Actually, various domains $\tilde{G}(y)$ are acceptable for the correct parametrization of spinor spaces, and one may choose any of them.

2.10. Spatial spinors in spherical coordinates

We consider the system of spherical coordinates

$$\begin{aligned} x_1 &= y_1 \sin y_2 \cos y_3, & x_2 &= y_1 \sin y_2 \sin y_3, & x_3 &= y_1 \cos y_2, \\ y_1 &\in [0, +\infty), & y_2 &\in [0, +\pi], & y_3 &\in [0, +2\pi]. \end{aligned} \tag{2.28}$$

A spinor $\eta(y)$ of the pseudo-vector model Π_3 is given by

$$\xi = \begin{pmatrix} \sqrt{y_1(1 + \cos y_2)} e^{-iy_3/2} \\ \sqrt{y_1(1 - \cos y_2)} e^{+iy_3/2} \end{pmatrix}. \tag{2.29}$$

In turn, a spinor $\eta(y)$ of the proper vector model E_3 is defined according to

$$\eta = \begin{pmatrix} \sqrt{y_1(1 - \sin y_2)} (\sigma e^{-iy_3/2}) \\ \sqrt{y_1(1 + \sin y_2)} e^{+iy_3/2} \end{pmatrix}, \tag{2.30}$$

The discontinuity properties of these spinors may be characterized by the diagram

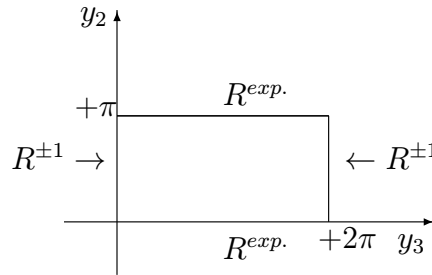


Fig. 7. Spherical coordinate / vector space

Evidently, the transition to extended models can be performed by formal doubling the range of angle variable y_3 ⁷

$$\tilde{G}(r, \theta, \phi) = \{ r \in [0, +\infty), \theta \in [0, +\pi], \phi \in [-2\pi, +2\pi] \}.$$

There are possible some alternative variants for the extended domain \tilde{G} , which can be used for covering spinor spaces. For instance, the most natural and symmetrical manner to do this, is to extend the range of radial variable:

$$\begin{aligned} \tilde{G}'(r, \theta, \phi) &= \{ r \in (-\infty, +\infty), \theta \in [0, +\pi], \phi \in [-\pi, -\pi] \}. \\ \xi(r, \theta, \phi) &= \begin{pmatrix} \sqrt{1 + \cos \theta} (\sqrt{r} e^{i\phi})^* \\ \sqrt{1 - \cos \theta} (\sqrt{r} e^{i\phi}) \end{pmatrix}. \end{aligned} \tag{2.31}$$

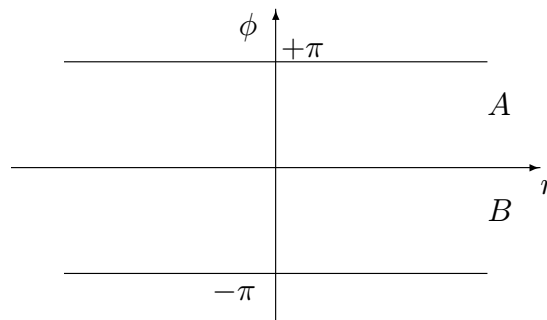


Fig. 8. Spherical coordinates / spinor space

⁷In the following we will use the more common notation $y_1 = r, y_2 = \theta, y_3 = \phi$

2.11. Conclusion

The results obtained for the 3-space with (x, y, z) coordinates may be extended to Minkowski 4-space with coordinates (t, x, y, z) . Mathematically, this means to use the relativistic $SL(2, \mathbb{C})$ spinors instead of non-relativistic $SU(2)$ spinors. The domains of curvilinear coordinates associated to spinor spaces can be used in order to examine possible quantum mechanical manifestation of the spinor structure, both in non-relativistic and relativistic theories. To this end, one should specially examine the analytical properties of the known solutions of the Schrödinger and Dirac equations in various coordinates.

3 Space with spinor structure and analytical properties of the solutions of the Klein–Fock–Gordon equation in cylindric parabolic coordinates

3.1. Parabolic cylindrical coordinates

Let us start with the cylindric parabolic coordinates

$$x = (u^2 - v^2)/2, \quad y = u v, \quad z = z. \quad (3.1)$$

In order to cover the vector space (x, y, z) , it suffices to make a choice out of the four possibilities:

$$\begin{aligned} v &= +\sqrt{-x + \sqrt{x^2 + y^2}}, & u &= \pm\sqrt{+x + \sqrt{x^2 + y^2}}, \\ v &= -\sqrt{-x + \sqrt{x^2 + y^2}}, & u &= \pm\sqrt{+x + \sqrt{x^2 + y^2}}, \\ v &= \pm\sqrt{-x + \sqrt{x^2 + y^2}}, & u &= +\sqrt{+x + \sqrt{x^2 + y^2}}, \\ v &= \pm\sqrt{-x + \sqrt{x^2 + y^2}}, & u &= -\sqrt{+x + \sqrt{x^2 + y^2}}. \end{aligned} \quad (3.2)$$

For definiteness, let us use the first variant from the above:

$$v = +\sqrt{-x + \sqrt{x^2 + y^2}}, \quad u = \pm\sqrt{+x + \sqrt{x^2 + y^2}}. \quad (3.3)$$

which is illustrated in Figure 9.

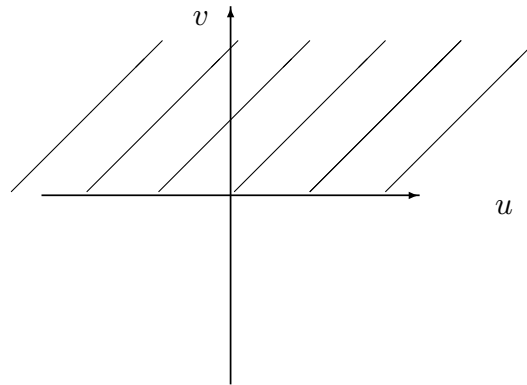


Fig. 9. The domain $G(u, v)$ used to parameterize the vector model

The correspondence between the points (x, y) and (u, v) can be illustrated by the following formulas and Figure 10:

$$\begin{aligned} u &= k \cos \phi, & v &= k \sin \phi, & \phi &\in [0, \pi]; \\ x &= (k^2/2) \cos 2\phi, & y &= (k^2/2) \sin 2\phi, & 2\phi &\in [0, 2\pi]. \end{aligned} \quad (3.4)$$

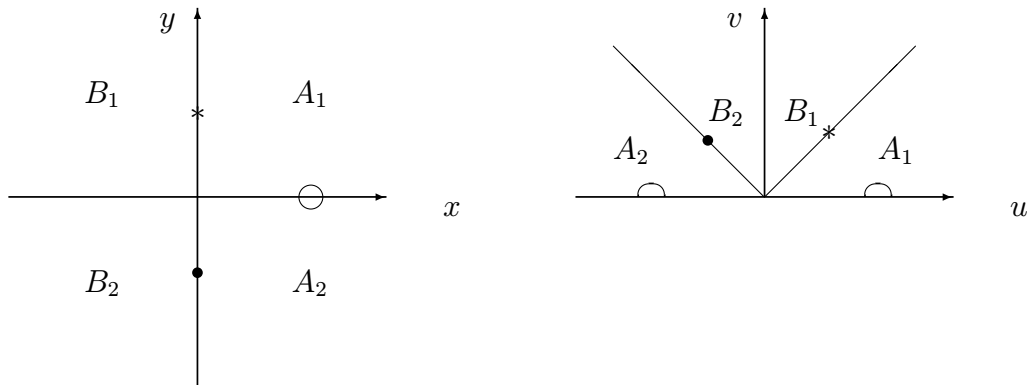


Fig. 10. The mapping $G(x, y) \implies G(u, v)$; identification rules

When turning to the case of spinor space, we will see the complete symmetry between the coordinates u and v ; they relate to the Cartesian coordinates of the extended model $(x, y, z) \oplus (x', y', z')$ through the formulas

$$v = \pm \sqrt{-x + \sqrt{x^2 + y^2}}, \quad u = \pm \sqrt{+x + \sqrt{x^2 + y^2}}, \tag{3.5}$$

illustrated by Figure 11:

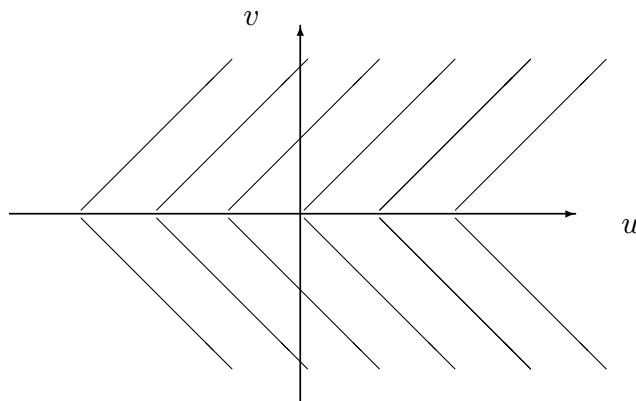


Fig. 11. $\tilde{G}(u, v)$ covering the spinor space

The metric of space-time in parabolic cylindrical coordinates has the form

$$dS^2 = c^2 dt^2 - (u^2 + v^2)(du^2 + dv^2) - dz^2. \tag{3.6}$$

3.1 3.2. Solutions of the Klein–Fock–Gordon equation and functions on the parabolic cylinder

Let us consider the KFG equation

$$\left[-\frac{1}{c^2} \frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial z^2} + \frac{1}{u^2 + v^2} \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right) - \frac{m^2 c^2}{\hbar^2} \right] \Psi = 0. \tag{3.7}$$

After separating the variables by the substitution

$$\Psi(t, u, v, \phi) = e^{-i\epsilon t/\hbar} e^{ipz/\hbar} U(u) V(v),$$

one gets

$$\left[\frac{1}{U} \frac{d^2 U}{du^2} + \left(\frac{\epsilon^2}{\hbar^2 c^2} - \frac{m^2 c^2}{\hbar^2} - \frac{p^2}{\hbar^2} \right) u^2 \right] + \left[\frac{1}{V} \frac{d^2 V}{dv^2} + \left(\frac{\epsilon^2}{\hbar^2 c^2} - \frac{m^2 c^2}{\hbar^2} - \frac{p^2}{\hbar^2} \right) v^2 \right] = 0. \quad (3.8)$$

In the following, we shall use the notation

$$\lambda^2 = \left(\frac{\epsilon^2}{\hbar^2 c^2} - \frac{m^2 c^2}{\hbar^2} - \frac{p^2}{\hbar^2} \right), \quad [\lambda] = \frac{1}{\text{meter}}. \quad (3.9)$$

By introducing two separation constants, a and b ($a + b = 0$), we can derive from (3.8) two distinct equations:

$$\frac{d^2 U}{du^2} + (\lambda^2 u^2 - a) U = 0, \quad \frac{d^2 V}{dv^2} + (\lambda^2 v^2 - b) V = 0. \quad (3.10)$$

The transition in equations (3.10) to the canonical form is obtained by using dimensionless variables:

$$\sqrt{2\lambda} u \rightarrow u, \quad \frac{a}{2\lambda} \rightarrow a, \quad \sqrt{2\lambda} v \rightarrow v, \quad \frac{b}{2\lambda} \rightarrow b. \quad (3.11)$$

The equations (3.10) will take the form:

$$\frac{d^2 U}{du^2} + \left(\frac{u^2}{4} - a \right) U = 0, \quad \frac{d^2 V}{dv^2} + \left(\frac{v^2}{4} + a \right) V = 0. \quad (3.12)$$

The solutions of these similar equations can be found as series:

$$F(\xi) = c_0 + c_1 \xi + c_2 \xi^2 + \sum_{k=1,2,\dots} c_{2k+1} \xi^{2k+1} + \sum_{k=1,2,\dots} c_{2k+2} \xi^{2k+2}; \quad (3.13)$$

we note that in (3.13) the terms of even and odd powers of ξ are distinguished.

After tedious calculation, one derives two independent groups of recurrent relations:

for even powers

$$\begin{aligned} \xi^0 : & \quad 2 c_2 - \alpha c_0 = 0, \\ \xi^2 : & \quad c_4 4 \times 3 + \frac{c_0}{4} - \alpha c_2 = 0, \\ \xi^4 : & \quad c_6 6 \times 5 + \frac{c_2}{4} - \alpha c_4 = 0, \\ n = 3, 4, \dots, \xi^{2n} : & \quad c_{2n+2}(2n+2)(2n+1) + \frac{1}{4} c_{2n-2} - \alpha c_{2n} = 0; \end{aligned} \quad (3.14)$$

for odd powers

$$\begin{aligned} \xi^1 : & \quad c_3 3 \times 2 - \alpha c_1 = 0, \\ \xi^3 : & \quad c_5 5 \times 4 + \frac{c_1}{4} - \alpha c_3 = 0, \\ n = 3, 4, \dots, \xi^{2n-1} : & \quad c_{2n+1}(2n+1)(2n) + \frac{1}{4} c_{2n-3} - \alpha c_{2n-1} = 0. \end{aligned} \quad (3.15)$$

So one can construct two linearly independent solutions

even

$$\begin{aligned}
 F_1(\xi^2) &= 1 + a_2 \frac{\xi^2}{2!} + a_4 \frac{\xi^4}{4!} + \dots, \\
 a_2 &= \alpha, \quad a_4 = \alpha^2 - \frac{1}{2}, \quad c_6 = \alpha^3 - \frac{7}{2}\alpha, \\
 n = 3, 4, \dots : \quad a_{2n+2} &= \alpha a_{2n} - \frac{(2n)(2n-1)}{4} a_{2n-2};
 \end{aligned}
 \tag{3.16}$$

odd

$$\begin{aligned}
 F_2(\xi) &= \xi + a_3 \frac{\xi^3}{3!} + a_5 \frac{\xi^5}{5!} + \dots, \\
 a_3 &= \alpha, \quad a_5 = \alpha^2 - \frac{3}{2}, \\
 n = 3, 4, \dots : \quad a_{2n+1} &= \alpha a_{2n-1} - \frac{(2n-1)(2n-2)}{4} a_{2n-3}.
 \end{aligned}
 \tag{3.17}$$

3.3. The basis wave functions. Manifestation of vector and spinor space structures

Having combined the two previous solutions F_1 and F_2 , we can obtain four types of wave functions⁸

$$\begin{aligned}
 (\text{even} \otimes \text{even}) : \quad \Phi_{++} &= E(a, u^2) E(-a, v^2), \\
 (\text{odd} \otimes \text{odd}) : \quad \Phi_{--} &= O(a, u) O(-a, v), \\
 (\text{even} \otimes \text{odd}) : \quad \Phi_{+-} &= E(a, u^2) O(-a, v), \\
 (\text{odd} \otimes \text{even}) : \quad \Phi_{-+} &= O(a, u) E(-a, v^2).
 \end{aligned}
 \tag{3.18}$$

Note the behavior of the constructed wave functions:

$$\begin{aligned}
 \Phi_{++}(x = 0, y = 0) &\neq 0, \quad \Phi_{--}(x = 0, y = 0) = 0, \\
 \Phi_{+-}(x > 0, y = 0) &= 0, \quad \Phi_{-+}(x < 0, y = 0) = 0.
 \end{aligned}
 \tag{3.19}$$

Now let us consider which restrictions for the wave functions Ψ follow from the requirement of single-valuedness. Here two peculiarities of the parametrization are substantial:

$$\underline{v = 0} : x = +\frac{u^2}{2} \geq 0, y = 0; \quad \underline{u = 0} : x = -\frac{v^2}{2} \leq 0, y = 0.
 \tag{3.20}$$

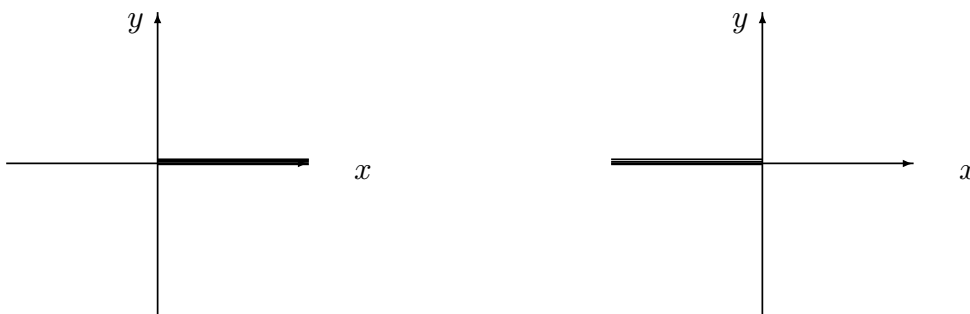


Fig. 12. The peculiarities of the parametrization

⁸We will change the notation: $F_1 \implies E$; $F_2 \implies O$.

Four solutions behave in special peculiar regions, as follows:

$$\Phi_{++}(a; u = 0, v) = + \Phi_{++}(a; u = 0, -v) , \tag{3.21}$$

$$\Phi_{++}(a; +u, v = 0) = + \Phi_{++}(a; -u, v = 0) , \tag{3.22}$$

$$\Phi_{--}(a; u = 0, +v) = + \Phi_{--}(a; u = 0, -v) = 0 \tag{3.23}$$

$$\Phi_{--}(a; u, v = 0) = + \Phi_{--}(a; -u, v = 0) = 0 , \tag{3.24}$$

$$\Phi_{+-}(a; u = 0, +v) = - \Phi_{+-}(a; u = 0, -v) , \tag{3.25}$$

$$\Phi_{+-}(a; u, v = 0) = \Phi_{+-}(a; -u, v = 0) = 0 , \tag{3.26}$$

$$\Phi_{-+}(a; u = 0, +v) = \Phi_{-+}(a; u = 0, -v) = 0 , \tag{3.27}$$

$$\Phi_{-+}(a; +u, v = 0) = - \Phi_{-+}(a; -u, v = 0) . \tag{3.28}$$

The boundary properties of the constructed wave functions can be illustrated by the following schemes:

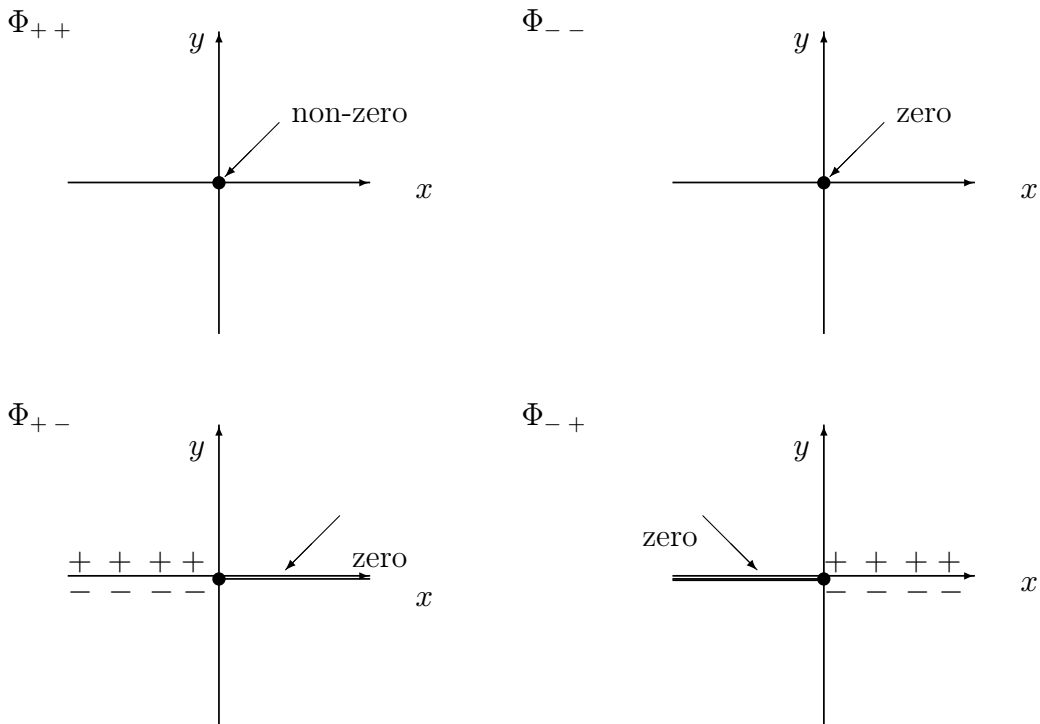


Fig 13. Boundary behavior of the wave functions in the (x, y) -plane

So we conclude that the solutions Ψ of the types $(++)$ and $(--)$ are single-valued in the space with vector structure, whereas the solutions of the types $(+-)$ and $(-+)$ are not single-valued in such a space, so these latter types $(+-)$ and $(-+)$ must be discarded. However, these solutions $((+-)$ and $(-+)$) must be retained in the space with *spinor* structure.

When using the spinor space model, two sets (u, v) and $(-u, -v)$ represent different geometrical points in the spinor space, so the requirement of single valuedness as applied in the case of spinor space does not assume that the values of the wave functions must be equal at the points (u, v) and $(-u, -v)$:

$$\Phi(u, v) = \Phi(x, y) \neq \Phi(-u, -v) = \Phi(x', y') . \tag{3.29}$$

The dividing of the basis wave functions into two subsets may be mathematically formalized with the help of the special discrete operator acting in the spinor space:

$$\hat{\delta} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \hat{\delta} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -u \\ -v \end{pmatrix}. \quad (3.30)$$

It is easily verified that the solutions which are single-valued in the vector space model are eigenfunctions of δ for the eigenvalue $\delta = +1$:

$$\hat{\delta} \Phi_{++}(a; u, v) = + \Phi_{++}(a; u, v), \quad (3.31)$$

$$\hat{\delta} \Phi_{--}(a; u, v) = + \Phi_{--}(a; u, v), \quad (3.32)$$

and the additional ones - which are acceptable only in the spinor space model - are eigenfunctions for the eigenvalue $\delta = -1$:

$$\hat{\delta} \Phi_{+-}(a; u, v) = - \Phi_{+-}(a; u, v), \quad (3.33)$$

$$\hat{\delta} \Phi_{-+}(a; u, v) = - \Phi_{-+}(a; u, v). \quad (3.34)$$

3.4. The form of a diagonalized operator \hat{A}

Let us find an explicit form of the operator \hat{A} , introduced above by the equation $\hat{A}\Psi = a \Psi$.

In Cartesian coordinates one has the following representation

$$\hat{A} = x \left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} \right) + 2y \frac{\partial^2}{\partial x \partial y} + \frac{\partial}{\partial x} + x \left(-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial z^2} - m^2 \right); \quad (3.35)$$

which in (u, v, z) -coordinates has the form

$$\hat{A} = \frac{1}{2} \left[\left(\frac{\partial^2}{\partial u^2} - \frac{\partial^2}{\partial v^2} \right) - \left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial z^2} + m^2 \right) (u^2 - v^2) \right]. \quad (3.36)$$

3.5. Orthogonality and completeness of the bases for vector and spinor spaces

Now let us consider the scalar multiplication

$$\int \Psi_{\mu'}^* \Psi_{\mu} \sqrt{-g} dt dz du dv. \quad (3.37)$$

of the basic constructed wave functions:

$$\Psi_{++}(\epsilon, p, a) = e^{i\epsilon t} e^{ipz} \Phi_{++}(a; u, v), \quad (3.38)$$

$$\Psi_{--}(\epsilon, p, a) = e^{i\epsilon t} e^{ipz} \Phi_{--}(a; u, v), \quad (3.39)$$

$$\Psi_{+-}(\epsilon, p, a) = e^{i\epsilon t} e^{ipz} \Phi_{+-}(a; u, v), \quad (3.40)$$

$$\Psi_{-+}(\epsilon, p, a) = e^{i\epsilon t} e^{ipz} \Phi_{-+}(a; u, v). \quad (3.41)$$

where μ and μ' stand for generalized quantum numbers.

First of all, we note some interesting integrals⁹:

in vector space

$$I_0 = \int_0^{+\infty} dv \int_{-\infty}^{+\infty} du \Phi_{++}^* \Phi_{--} (u^2 + v^2), \quad (3.42)$$

⁹The arguments $(a; u, v)$ are omitted here.

in spinor space

$$I_1 = \int_{-\infty}^{+\infty} dv \int_{-\infty}^{+\infty} du \Phi_{++}^* \Phi_{--} (u^2 + v^2) , \tag{3.43}$$

$$I_2 = \int_{-\infty}^{+\infty} dv \int_{-\infty}^{+\infty} du \Phi_{+-}^* \Phi_{-+} (u^2 + v^2) , \tag{3.44}$$

$$I_3 = \int_{-\infty}^{+\infty} dv \int_{-\infty}^{+\infty} du \Phi_{++}^* \Phi_{+-} (u^2 + v^2) , \tag{3.45}$$

$$I_4 = \int_{-\infty}^{+\infty} dv \int_{-\infty}^{+\infty} du \Phi_{++}^* \Phi_{-+} (u^2 + v^2) , \tag{3.46}$$

$$I_5 = \int_{-\infty}^{+\infty} dv \int_{-\infty}^{+\infty} du \Phi_{--}^* \Phi_{+-} (u^2 + v^2) , \tag{3.47}$$

$$I_6 = \int_{-\infty}^{+\infty} dv \int_{-\infty}^{+\infty} du \Phi_{--}^* \Phi_{-+} (u^2 + v^2) . \tag{3.48}$$

All these seven integrals $I_0, I_1 \dots I_6$ are equal to zero, which means that the constructed functions provide us with an orthogonal basis for the Hilbert space $\Psi(t, z, u, v)$, where (u, v, z) belong to the extended (spinor) space model.

3.6. On matrix elements of physical observables, in vector and spinor spaces

The question of principle is to determine in which way the transition from vector to spinor space model can influence the results of calculation of matrix elements for physical quantities. As an example, let us consider matrix elements for operator of coordinates: one may calculate the matrix elements of the basic initial coordinates u, v or x, y :

$$x = \frac{u^2 - v^2}{2} , \quad y = uv , \quad \text{or} \quad (u, v) . \tag{3.49}$$

Then simple selection rules for the matrix elements can be derived¹⁰:

in vector space

$$\begin{array}{cc} x_{\mu',\mu} & ++ \quad -- \\ ++ & \neq 0 \quad 0 \\ -- & 0 \quad \neq 0 \end{array} , \quad \begin{array}{cc} y_{\mu',\mu} & ++ \quad -- \\ ++ & 0 \quad \neq 0 \\ -- & \neq 0 \quad 0 \end{array} \tag{3.50}$$

in spinor space

$$\begin{array}{cccc} x_{\mu',\mu} & ++ & -- & +- & -+ \\ ++ & \neq 0 & 0 & 0 & 0 \\ -- & 0 & \neq 0 & 0 & 0 \\ +- & 0 & 0 & \neq 0 & 0 \\ -+ & 0 & 0 & 0 & \neq 0 \end{array} , \quad \begin{array}{cccc} y_{\mu',\mu} & ++ & -- & +- & -+ \\ ++ & 0 & \neq 0 & 0 & 0 \\ -- & \neq 0 & 0 & 0 & 0 \\ +- & 0 & 0 & 0 & \neq 0 \\ -+ & 0 & 0 & \neq 0 & 0 \end{array} \tag{3.51}$$

The same, for the coordinates u and v , looks like:

¹⁰For simplicity we restrict ourselves only to the degeneracy at the discrete quantum number $++$, $--$, $+-$, $-+$, by taking ϵ, p, a fixed.

in vector space

$$\begin{array}{ccc}
 \underline{u_{\mu',\mu}} & ++ & -- \\
 ++ & 0 & \neq 0 \\
 -- & \neq 0 & 0
 \end{array}
 , ,
 \begin{array}{ccc}
 \underline{v_{\mu',\mu}} & ++ & -- \\
 ++ & \neq 0 & 0 \\
 -- & 0 & \neq 0
 \end{array}
 \tag{3.52}$$

in spinor space

$$\begin{array}{cccc}
 \underline{u_{\mu',\mu}} & ++ & -- & +- & -+ \\
 ++ & 0 & 0 & 0 & \neq 0 \\
 -- & 0 & 0 & \neq 0 & 0 \\
 +- & 0 & \neq 0 & 0 & 0 \\
 -+ & \neq 0 & 0 & 0 & 0
 \end{array}
 ,
 \begin{array}{cccc}
 \underline{v_{\mu',\mu}} & ++ & -- & +- & -+ \\
 ++ & 0 & 0 & \neq 0 & 0 \\
 -- & 0 & 0 & 0 & \neq 0 \\
 +- & \neq 0 & 0 & 0 & 0 \\
 -+ & 0 & \neq 0 & 0 & 0
 \end{array}
 \tag{3.53}$$

3.7. Schrödinger equation

The study of the analytical properties of the Klein-Fock-Gordon wave solutions in vector and spinor space models is still applicable, with slight changes, to the non-relativistic Schrödinger equation as well:

$$i\hbar \frac{\partial}{\partial t} \Psi = -\frac{\hbar^2}{2m} \left[\frac{\partial^2}{\partial z^2} + \frac{1}{u^2 + v^2} \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right) \right] \Psi , \tag{3.54}$$

where the substitution for the wave functions is the same

$$\Psi(t, u, v, z) = e^{-iet/\hbar} e^{ipz/\hbar} U(u)V(v) , \tag{3.55}$$

and then, the equation for $U(u)V(v)$ is

$$\left[\frac{\hbar^2}{2m} \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right) + \left(\epsilon - \frac{p^2}{2m} \right) (u^2 + v^2) \right] U(u)V(v) = 0 . \tag{3.56}$$

3.8. Results

We shall further infer several quantum mechanical consequences while changing the vector geometrical model of the physical space to the spinor one.

The extension procedure is performed in cylindrical parabolic coordinates, $G(t, u, v, z) \implies \tilde{G}(t, u, v, z)$. This is done through expansion twice as much of the domain G , so that instead of the half plane ($u, v > 0$) now the entire plane (u, v) should be used, accompanied with new identification rules for the boundary points. In the Cartesian picture, this procedure corresponds to taking the two-sheet surface $(x', y') \oplus (x'', y'')$ instead of the one-sheet surface (x, y) .

The solutions of the Klein-Fock-Gordon and Schrödinger equations $\Psi_{\epsilon, p, a} = e^{iet} e^{ipz} U_a(u) V_a(v)$ are constructed in terms of parabolic cylindrical functions¹¹. Given the quantum numbers ϵ, p, a , four types of solutions are possible: $\Psi_{++}, \Psi_{--}; \Psi_{+-}, \Psi_{-+}$.

The first two ones, Ψ_{++} and Ψ_{--} , provide us with single-valued functions of the vector space points, whereas the last two, Ψ_{+-} and Ψ_{-+} , have discontinuities in the frame of vector spaces, and therefore they must be discarded in this model. All the four types of functions are continuous ones while regarded in the spinor space.

¹¹We denoted the separating constant by a .

It is established that all solutions Ψ_{++} , Ψ_{--} , Ψ_{+-} and Ψ_{-+} , are orthogonal to each other, provided that integration is done over the extended domain of parametrization of the spinor space.

Some simple selection rules for matrix elements of the vector and spinor coordinates, (x, y) and (u, v) , respectively, are further derived. The selection rules for (u, v) are substantially different in vector spaces compared to spinor spaces.

4 Some relevant topics

In conclusion, within this paper we addressed the following topics: the spinor structure of space models; the relation between the Dirac–Schwinger quantization rule and the superposition principle in quantum mechanics; the manifestation of spinor space structure in classifying the solutions of the Dirac equation and for the matrix elements which are related to physical quantities; spinors in polarization optics; the Jones formalism for completely and partly polarized light; General Relativity and Riemannian space-time models with spinor structure and tetrad (vierbein) formalism.

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О ВОЗМОЖНЫХ ПРОЯВЛЕНИЯХ СПИНОРНЫХ СТРУКТУР В КВАНТОВОЙ ФИЗИКЕ

Елена Овсюк¹, Ольга Веко¹, Алесандру Оана², Мирче Неагу²,
Владимир Балан³, Виктор Редьков⁴

¹ Мозырский Государственный Педагогический Университет, Мозырь, Беларусь

² Брашовский Трансильванский Университет, Брашов, Румыния

³ Бухарестский Политехнический Университет, Бухарест, Румыния

⁴ Институт Физики им. Б.И. Степанова, Национальная Академия Наук Беларуси, Минск,
Беларусь

v.redkov@dragon.bas-net.by

В работе обсуждаются следующие вопросы: спинорная накрывающая группы Лоренца, внутренняя четность фермиона, майорановские фермионы, модели пространств со спинорной структурой, параметризация спинорных пространств криволинейными координатами, проявление спинорной пространственной структуры в классификации решений квантово-механических уравнений и в матричных элементах физических величин

Ключевые слова: спиноры, группа Лоренца, четность релятивистского фермиона, 4-спиноры Майораны, пространство со спинорной структурой, уравнение Клейна-Фока-Гордона, квантовая механика