DUAL SPACES, PARTICLE SINGULARITIES AND QUARTIC GEOMETRY

Peter Rowlands
University of Liverpool, Liverpool, UK
p.rowlands@liverpool.ac.uk

Relativistic quantum mechanics and the properties of Dirac fermions can be generated in a particularly powerful way using two vector spaces which are commutative to each other and which contain identical information. The apparently broken symmetry between the two spaces observed through the quadratic geometry of ordinary space becomes a perfect and unbroken symmetry in the quartic geometry which defines the single physical quantity through which the two spaces can be combined.

Key Words: dual vector spaces, nilpotent quantum mechanics, vacuum space, fermion singularity, H4 algebra

1 Geometry and physics

Can we construct physics entirely in terms of geometry? This is a highly relevant question in view of the fact that our only way of apprehending Nature is through 3-dimensional space. Nothing else is directly observable. The attempt has been made many times – Cartesian philosophy general relativity, Kaluza-Klein theory, unified field theory, and string and membrane theories are examples - but never with completely satisfactory results. Whatever number of dimensions we add to our space-time structure, and however we contort it, we have to face the fact that the world we observe is 3-dimensional. Clearly something else is there, but we do not observe it. Can we find a geometrical way of constructing it, so that we create a larger geometry which still preserves the 3-dimensional nature of observation?

I am actually going to propose that the world is not 3-dimensional, or constructed from space of any other dimensionality. In fact, it has no structure whatsoever. It is a zero totality. To reconcile this with our view of a 3-dimensional spatial world, we have to imagine a dual 3-dimensional space, which in some subtle way cancels the effect of our observed space. Providing our known 3-dimensional Euclidean space with its dual partner allows us to construct an algebraic geometry which has a remarkable parallels to the one that seems to operate in the real world.

The only insight ever attained into the meaning of 3-dimensionality came with the discovery of quaternions. Here it is associated with anticommutativity. The four quaternion units, $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$, 1, follow the well-known multiplication rules:

$$\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -1 \tag{1}$$

$$\boldsymbol{ii} = -\boldsymbol{ji} = \boldsymbol{k} \tag{2}$$

$$\boldsymbol{jk} = -\boldsymbol{kj} = \boldsymbol{i} \tag{3}$$

$$\boldsymbol{ki} = -\boldsymbol{ik} = \boldsymbol{j}. \tag{4}$$

If we make our units anticommutative but still associative, we are obliged to fix them at 3.

We can of course complexify quaternions, to create a set of units, $(i\boldsymbol{i})$, $(i\boldsymbol{j})$, $(i\boldsymbol{k})$, i, which are the complexified versions of the quaternion ones. So $(i\boldsymbol{i}) = \mathbf{i}$, $(i\boldsymbol{j}) = \mathbf{j}$, $(i\boldsymbol{k}) = \mathbf{k}$, $(i1) = i$ The units follow the multiplication rules:

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{1} \tag{5}$$

$$\mathbf{ij} = -\mathbf{ji} = i\mathbf{k} \tag{6}$$

$$\mathbf{jk} = -\mathbf{kk} = i\mathbf{i} \tag{7}$$

$$\mathbf{ki} = -\mathbf{ik} = i\mathbf{j}. \tag{8}$$

The units, which are the complexified versions of the quaternion units, have acquired a number of names. They are called multivariate vectors by Hestenes [1], as they have all the properties of ordinary vectors, except that they also have a full (algebraic) product:

$$\mathbf{ab} = \mathbf{a} \cdot \mathbf{b} + i\mathbf{a} \times \mathbf{b} \tag{9}$$

from which all the rules concerning unit vector multiplication may be derived. They are isomorphic to Pauli matrices. In more general terms, however, they are the units of the Clifford (or geometrical) algebra of 3-dimensional space.

Of course, if we complexify the units of the multivariate or Clifford algebra, we revert to quaternions, so $i\mathbf{i} = \boldsymbol{i}$, $i\mathbf{j} = \boldsymbol{j}$, $i\mathbf{k} = \boldsymbol{k}$, etc. However, terms like $i\mathbf{i}$, $i\mathbf{j}$, $i\mathbf{k}$ are also recognisable as units of pseudovectors or axial vectors (e.g. area, angular momentum) and i is recognisable as a unit pseudoscalar (e.g. volume). All real vectors are of this type. There are no 'ordinary' vectors in nature.

2 A dual vector space

The units $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ define a complete Clifford algebra of 3D space:

$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$	*vector*		
$i\mathbf{i}$	$i\mathbf{j}$	$i\mathbf{k}$	*bivector*	*pseudovector*	*quaternion*
i			*trivector*	*pseudoscalar*	
1			*scalar*		

where the pseudovectors give us areas and the pseudoscalars volumes.

Let us suppose we have another such algebra, isomorphic with the first:

$\mathbf{I}$	$\mathbf{J}$	$\mathbf{K}$	*vector*		
$i\mathbf{I}$	$i\mathbf{J}$	$i\mathbf{K}$	*bivector*	*pseudovector*	*quaternion*
i			*trivector*	*pseudoscalar*	
1			*scalar*		

If we combine these two algebras commutatively in a tensor product, that is, we take the algebraic product of the eight base units, 1, $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, i, $\mathbf{I}$, $\mathbf{J}$, $\mathbf{K}$, we obtain 64 terms, which are $+$ and $-$ versions of:

$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$	$i\mathbf{i}$	$i\mathbf{j}$	$i\mathbf{k}$	i	1
$\mathbf{I}$	$\mathbf{J}$	$\mathbf{K}$	$i\mathbf{I}$	$i\mathbf{J}$	$i\mathbf{K}$		
$\mathbf{iI}$	$\mathbf{jI}$	$\mathbf{kI}$	$i\mathbf{iI}$	$i\mathbf{jI}$	$i\mathbf{kI}$		
$\mathbf{iJ}$	$\mathbf{jJ}$	$\mathbf{kJ}$	$i\mathbf{iJ}$	$i\mathbf{jJ}$	$i\mathbf{kJ}$		
$\mathbf{iK}$	$\mathbf{jK}$	$\mathbf{kK}$	$i\mathbf{iK}$	$i\mathbf{jK}$	$i\mathbf{kK}$		

This becomes a double vector algebra or a double Clifford algebra of 3D space.

We can also take the algebraic product of the four quaternion units, 1, $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$,and the four vector units i, $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, to produce an exactly isomorphic vector quaternion algebra, whose units are $+$ and $-$ versions of:

$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$	$i\mathbf{i}$	$i\mathbf{j}$	$i\mathbf{k}$	i	1
$\boldsymbol{i}$	$\boldsymbol{j}$	$\boldsymbol{k}$	$i\boldsymbol{i}$	$i\boldsymbol{j}$	$i\boldsymbol{k}$		
$\mathbf{i}\boldsymbol{i}$	$\mathbf{j}\boldsymbol{i}$	$\mathbf{k}\boldsymbol{i}$	$i\mathbf{i}\boldsymbol{i}$	$i\mathbf{j}\boldsymbol{i}$	$i\mathbf{k}\boldsymbol{i}$		
$\mathbf{i}\boldsymbol{j}$	$\mathbf{j}\boldsymbol{j}$	$\mathbf{k}\boldsymbol{j}$	$i\mathbf{i}\boldsymbol{j}$	$i\mathbf{j}\boldsymbol{j}$	$i\mathbf{k}\boldsymbol{j}$		
$\mathbf{i}\boldsymbol{k}$	$\mathbf{j}\boldsymbol{k}$	$\mathbf{k}\boldsymbol{k}$	$i\mathbf{i}\boldsymbol{k}$	$i\mathbf{j}\boldsymbol{k}$	$i\mathbf{k}\boldsymbol{k}$		

Yet another isomorphic version of the same algebra appears when we take a complexified algebraic product of two commutative sets of quaternion units $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$, $\boldsymbol{I}$, $\boldsymbol{J}$, $\boldsymbol{K}$. This complexified double quaternion algebra has units which are + and − versions of:

$\boldsymbol{i}$	$\boldsymbol{j}$	$\boldsymbol{k}$	$i\boldsymbol{i}$	$i\boldsymbol{j}$	$i\boldsymbol{k}$	i	1
$\boldsymbol{I}$	$\boldsymbol{J}$	$\boldsymbol{K}$	$i\boldsymbol{I}$	$i\boldsymbol{J}$	$i\boldsymbol{K}$		
$\boldsymbol{iI}$	$\boldsymbol{jI}$	$\boldsymbol{kI}$	$i\boldsymbol{iI}$	$i\boldsymbol{jI}$	$i\boldsymbol{kI}$		
$\boldsymbol{iJ}$	$\boldsymbol{jJ}$	$\boldsymbol{kJ}$	$i\boldsymbol{iJ}$	$i\boldsymbol{jJ}$	$i\boldsymbol{kJ}$		
$\boldsymbol{iK}$	$\boldsymbol{jK}$	$\boldsymbol{kK}$	$i\boldsymbol{iK}$	$i\boldsymbol{jK}$	$i\boldsymbol{kK}$		

This dual vector space algebra is of immense physical significance, for it is isomorphic to the gamma algebra of the Dirac equation, which defines the relativistic quantum mechanics of the fermionic state. Though the gamma algebra is usually based on 4×4 matrices, all such matrices can, in fact, be derived from the products of two commuting sets of 2×2 Pauli matrices, say σ_1, σ_2, σ_3 and Σ_1, Σ_2, Σ_3. This is identical to deriving them from the units of two vector spaces: **i**, **j**, **k** and **I**, **J**, **K**. Relativistic quantum mechanics, it seems, requires a dual vector space, in addition to the 'doubling' produced by the complex nature of each vector space.

The units also form a group of order 64, with a minimum of 5 generators. The 5 generators of the algebraic group can be matched to the 5 gamma matrices in a number of ways. There are many ways of doing this but the key sets of generators include all the individual units of the two 3-dimensional quantities (vector / quaternion) or two 'spaces' and the overall structure of these sets is always the same. Because 5 is not a truly symmetrical number in nature, the symmetry of one of the two spaces is preserved (here, represented by lower case characters), while that of the other (here, represented by upper case characters) is broken:

$$\gamma_0 = i\boldsymbol{k}; \quad \gamma_1 = \boldsymbol{i}\mathbf{i}; \quad \gamma_2 = \boldsymbol{i}\mathbf{j}; \quad \gamma_3 = \boldsymbol{i}\mathbf{k}; \quad \gamma_5 = i\boldsymbol{j}.$$
$$\gamma_0 = \mathbf{k}; \quad \gamma_1 = i\mathbf{Ii}; \quad \gamma_2 = i\mathbf{Ij}; \quad \gamma_3 = i\mathbf{Ik}; \quad \gamma_5 = \mathbf{J}.$$

3 The H4 algebra

The 64-part algebra has a subalgebra which is particularly significant for physics, and this creates a symmetry between the two spaces which remains unbroken. The algebra can be constructed using coupled quaternions, with units 1, $\boldsymbol{iI}$, $\boldsymbol{jJ}$, $\boldsymbol{kK}$, to produce a cyclic but commutative algebra with multiplication rules:

$$\boldsymbol{iIiI} = \boldsymbol{jJjJ} = \boldsymbol{kKkK} = 1 \tag{10}$$

$$\boldsymbol{iIjJ} = \boldsymbol{jJiI} = \boldsymbol{kK} \tag{11}$$

$$\boldsymbol{jJkK} = \boldsymbol{kKjJ} = \boldsymbol{iI} \tag{12}$$

$$\boldsymbol{kKiI} = \boldsymbol{iIkK} = \boldsymbol{jJ} \tag{13}$$

Again, there are alternative ways of constructing the algebra, one of which uses the negative values of the paired vector units 1, $-\mathbf{iI}$, $-\mathbf{jJ}$, $-\mathbf{kK}$. (1 is equivalent here to $-ii$.) This time we have:

$$(-\mathbf{iI})(-\mathbf{iI}) = (-\mathbf{jJ})(-\mathbf{jJ}) = (-\mathbf{kK})(-\mathbf{kK}) = \mathbf{1} \tag{14}$$

$$(-\mathbf{iI})(-\mathbf{jJ}) = (-\mathbf{jJ})(-\mathbf{iI}) = (-\mathbf{kK}) \tag{15}$$

$$(-\mathbf{jJ})(-\mathbf{kK}) = (-\mathbf{kK})(-\mathbf{jJ}) = (-\mathbf{iI}) \tag{16}$$

$$(-\mathbf{kK})(-\mathbf{iI}) = (-\mathbf{iI})(-\mathbf{kK}) = (-\mathbf{jJ}) \tag{17}$$

Using the symbols $\mathrm{I} = \boldsymbol{iI} = -\mathbf{iI}$, $\mathrm{J} = \boldsymbol{jJ} = -\mathbf{jJ}$, $\mathrm{K} = \boldsymbol{kK} = -\mathbf{kK}$, 1, to represent the units of the algebra, we can structure the relationships between them in a group table:

*		1	I	J	K
1		1	I	J	K
I		I	1	K	J
J		J	K	1	I
K		K	I	J	1

This can be seen as a representation of the Klein-4 group, the noncyclic group of order 4.

4 Nilpotent quantum mechanics

The double vector algebra allows us to create relativistic quantum mechanics in a particularly efficient and streamlined way [2,3]. We can, for example, begin with Einstein's energy-momentum conservation equation (with $c = 1$)

$$E^2 - p^2 - m^2 = 0 \tag{18}$$

factorize directly using the algebra, using any of the three isomorphic representations of the units. Here we will use the combination of four quaternion units (1, $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$) and four multivariate vector units (i, $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$), noting the similarity to Penrose's twistors, their with four real or norm 1 components and four imaginary or norm вЂ“1 components [4,5].

The eight base units (1, $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$, i, $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$) have a similar structure. There is a significant difference, however, in that the connection between the units of space and time now comes from a quantum rather than a classically relativistic structure. In effect, because of the mediating gamma matrices or algebraic operators, which are different for the space and time components, the space-time connection is now no longer purely 4-vector. Using the vector-quaternion algebra, we now factorize (18) in the form

$$(i\boldsymbol{k}E + \boldsymbol{i}\mathbf{i}p_x + \boldsymbol{i}\mathbf{j}p_y + \boldsymbol{i}\mathbf{k}p_z + \boldsymbol{j}m)(i\boldsymbol{k}E + \boldsymbol{i}\mathbf{i}p_x + \boldsymbol{i}\mathbf{j}p_y + \boldsymbol{i}\mathbf{k}p_z + \boldsymbol{j}m) = \mathbf{0} \tag{19}$$

or

$$(i\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)(i\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m) = \mathbf{0}. \tag{20}$$

The object $(i\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$ is a nilpotent, a square root of 0. It can be used to produce a powerful form of relativistic quantum mechanics. If we apply a canonical quantization procedure to the first such expression in equation (20), to replace the terms E and $\mathbf{p}$ by the operators $E \to i\partial/\partial t$, $\mathbf{p} \to -i\nabla$ (using units where $\hbar = 1$), and assume that the operators act on the phase factor for a free fermion, $e^{-i(Et-\mathbf{p.r})}$, we immediately obtain the nilpotent Dirac equation for a free fermion:

$$\left(\mp\boldsymbol{k}\frac{\partial}{\partial t} \mp i\boldsymbol{i}\nabla + \boldsymbol{j}m\right)(\pm i\boldsymbol{k}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)\mathbf{e}^{-i(Et-\mathbf{p.r})} = \mathbf{0} \tag{21}$$

As Hestenes has shown [1], spin is automatically included when we take $\mathbf{p}$ or ∇ as a multivariate vector through the extra $\times$ term in the full product. This means that we can interchange $\mathbf{p}$ with $\sigma.\mathbf{p}$ and ∇ with $\sigma.\nabla$ in equations such as (20) and (21).

The nilpotent formalism are also derivable from the conventional Dirac equation by pre-multiplication by γ_5, and conversion of the gamma matrices to algebraic operators. All conventional results are accessible to both representations, but the nilpotent formalisms uncovers additional details which tend to lie hidden in the matrix representation. As usual, The nature of the four simultaneous solutions required for the wavefunction (2 for fermion / antifermion $\times$ 2 for spin up / spin down) is immediately apparent. In the new formalism, we replace the 4×4 matrix differential operator and column vector wavefunction with a row vector operator and a column vector wavefunction, each of which may be represented in abbreviated form by $(\pm i\boldsymbol{k}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$. With choice of sign convention, the four solutions now become:

$(i\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$	fermion	spin up
$(i\boldsymbol{k}E - \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$	fermion	spin down
$(-i\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$	antifermion	spin down
$(-i\boldsymbol{k}E - \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$	antifermion	spin up

The meaning of the four terms is now apparent. The first term in the column may represent the observed particle state, while the others become the accompanying vacuum states, or states into which the observed particle could transform by respective P, T and C transformations:

$$\begin{array}{lll} P & \boldsymbol{i}(i\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)\boldsymbol{i} & = (i\boldsymbol{k}E - \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m) \\ T & \boldsymbol{k}(i\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)\boldsymbol{k} & = (-i\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m) \\ C & -\boldsymbol{j}(i\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)\boldsymbol{j} & = (-i\boldsymbol{k}E - \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m) \end{array}$$

If we replace the observed fermion state spin up by any of the others using any of the transformations P, T or C we will simultaneously apply the same transformation to all the others. Since specifying the first term necessarily specifies all the others, it is often convenient to write the 4-component wavefunction as a single term and assume the sign changes follow automatically.

To demonstrate the relation between three additional states on the fermion, the P, T, C transformations, and vacuum we can take $(\pm i\boldsymbol{k}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$ and post-multiply it by the idempotent $\boldsymbol{k}(\pm i\boldsymbol{k}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$ any number of times. Since the only effect of this operation is to introduce a scalar multiple, which can be normalized away, then the idempotent $\boldsymbol{k}(\pm i\boldsymbol{k}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$ is clearly acting as a vacuum operator.

$$(\pm i\boldsymbol{k}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)\boldsymbol{k}(\pm i\boldsymbol{k}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)\boldsymbol{k}(\pm i\boldsymbol{k}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m) \ldots \rightarrow (\pm i\boldsymbol{k}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m) \quad (22)$$

The same occurs with with $\boldsymbol{i}(\pm i\boldsymbol{k}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$ and $\boldsymbol{i}\boldsymbol{k}(\pm i\boldsymbol{k}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$, the extra vector terms in the first case being absorbed into the scalar multiple and disappearing with alternate multiplications. In effect, $\boldsymbol{k}(\pm i\boldsymbol{k}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$, $\boldsymbol{i}(\pm i\boldsymbol{k}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$ and $\boldsymbol{j}\boldsymbol{k}(\pm i\boldsymbol{k}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$ can be regarded as vacuum operators, and $\boldsymbol{k}$, $\boldsymbol{i}$ and $\boldsymbol{j}$, or, equivalently, $\mathbf{K}$, $\mathbf{I}$ and $\mathbf{J}$, as coefficients of a 'vacuum space'.

The most important consequence of adopting the nilpotent form of quantum mechanics, however, is that it produces an extra constraint which carries a wealth of new information about quantum systems, while reducing the information input required to a minimum. For example, an operator of the form $(i\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$ will automatically generate the phase term on which it operates to produce a nilpotent amplitude of the form $(i\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$, or one that squares to zero. This completely eliminates the need for an equation. Only the operator is required. Also, though equation (21) is specified for a free the fermion, the fermion need not be free. The method is equally valid when we incorporate field terms or covariant derivatives into the operator, for example, when we make the replacements $E \rightarrow i\partial/\partial t + e\phi + \ldots$, and $\mathbf{p} \rightarrow -i\nabla + e\mathbf{A} + \ldots$. The operator here still retains the overall form $(i\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$, but the phase term is now no longer be $e^{-i(Et-\mathbf{p}.\mathbf{r})}$, but whatever is needed to create an amplitude of the general form $(\boldsymbol{ik}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$, which squares to zero, and the eigenvalues E and $\mathbf{p}$ in the amplitude will be more complicated expressions resulting from the presence of the field terms.

In principle, space and time are completely variable as long as we preserve the conservation principles which define a system. The phase factor determines the extent to which this happens. In the case of a completely free fermion, there is no restriction on spatial position over any time period, and this is reflected in the phase factor $e^{-i(Et-\mathbf{p.r})}$, because there is no restraining conservation principle (except that of charge). As soon as the fermion interacts with another system, however, say another fermion, conservation principles will be invoked (leading to terms added to the space and time differentials) and this will be reflected in a more complicated phase factor, with a more restricted range of spatial variation. The restrictions will increase as more fermions are brought within range of the originally free fermion, and so the range of variation in the phase factor will decrease even further. The classical limit will be reached when it shows virtually no variation with space over time. The nilpotent formalism makes the quantum / classical transition simply a consequence of the number of conservation principles that have to be applied.

Many new results also emerge from the nilpotent formalism. Particularly important are the structures of the three different boson-type states, considered as combinations of an original fermion state with any of the P, T or C transformed ones, the result being a scalar wavefunction.

$(\pm \boldsymbol{ik}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)(\mp \boldsymbol{ik}\mathbf{E} \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$	spin 1 boson
$(\pm \boldsymbol{ik}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)(\mp \boldsymbol{ik}\mathbf{E} \mp \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$	spin 0 boson
$(\pm \boldsymbol{ik}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)(\pm \boldsymbol{ik}\mathbf{E} \mp \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$	fermion-fermion combination

Significantly, a spin 1 boson can be massless, but a non-vanishing spin 0 boson must have a mass, as the massless $(\pm \boldsymbol{ik}E \pm \boldsymbol{i}\mathbf{p})(\mp \boldsymbol{ik}\mathbf{E} \mp \boldsymbol{i}\mathbf{p})$ would immediately become zero. So, massless Goldstone bosons must become massive Higgs bosons in the Higgs mechanism.

In principle, the nilpotent formalism defines the relativistic quantum mechanics for a fermion in any state, subject to any number of interactions, simply by creating an operator of the form $(\pm \boldsymbol{ik}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$, which then uniquely determines the phase factor which will make the amplitude nilpotent.

$$\text{operator acting on phase factor}^2 = \text{amplitude}^2 = 0. \tag{23}$$

This can be accomplished without defining any equation at all.

5 The fermion as singularity

But where does this formalism come from? Space is a nonconserved quantity, and so cannot define on its own a point. Two 'spaces' are the minimum needed to define a particle singularity. While mathematicians may discuss points in ordinary 3-D space, physically they have no meaning, as space is a nonconserved quantity whose units have no definable identity because they have translation and rotation symmetry. In effect, we cannot identify anything in a single space, but identification becomes possible if we have two spaces.

We can here apply a reverse argument from topology. The creation of a particle singularity using its intersection with a dual space can be seen as the creation of a multiply-connected space from a simply-connected space through the insertion of a topological singularity. For fermions, we can describe one of these as real space and the other as the 'vacuum space' which we have previously defined. This space is closely connected with charge and the weak, strong and electric interactions, as well as T, P and c transformations. In this sense, we can say that the fermion always exists in the two spaces from which it is constructed, real space and vacuum space, and the non-classical *zitterbewgung* motion, which Schredinger found in the solution to the free-particle Dirac equation, represents the switching between these spaces which makes it possible to define the fermion as creating a point singularity through the intersection of two spaces.

The creation of a singularity using these two spaces determines that they are precisely dual and that each contains the same information as the other, though in a different form as regards observation. But as observers *within* the system, we are forced to 'privilege' one space over the other, to maintain the symmetry of one while losing that of the other. It is similar to the way in which our most primitive form of numbering, binary arithmetic, 'privileges' 1 over вЋ“1, making them dual in summing to 0, but appearing very different in the way they are perceived from within a system defined by unit 1.

An asymmetry or chirality appears in the fermionic structure because it results from an asymmetric combination of the space of observation with an unobserved dual vacuum space. Perfect symmetry would have been preserved if we had used 6 generators (**i**, **j**, **k**, **I**, **J**, **K**), but this is not the minimum. The minimum number of generators of the combined 64-part algebra is 5, a number intrinsically suggesting asymmetry, and this, as we have seen, requires the symmetry of one space to be broken while the other is preserved:

K	i**Ii** i**Ij** i**Ik**	i**J**
energy	*momentum*	*mass*
time	*space*	*proper time*

The space, in this formulation, with the unbroken symmetry (represented by lower case characters) is real space, the space of observation. The space with the broken symmetry (represented by upper case characters) is 'vacuum space', and it seems to be the space which combines all the unobservable quantities (specifically, time, mass, charge). The zeroing produced by the nilpotent condition ensures, as we can show, that the information in the two spaces represented by the respective units **i**, **j**, **k** and **I**, **J**, **K** is identical. It also defines, in principle, the meaning of a point in either of the two spaces as the norm 0 crossover between them.

Pauli exclusion, a fundamentally nonlocal phenomenon, is an immediate and unavoidable consequence of the fact that, in nilpotent quantum mechanics, the total structure of the universe is exactly zero. A fermion with a wavefunction of the form $\psi_f = (\boldsymbol{ik}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$ created from *absolutely nothing* requires the simultaneous existence of a dual 'vacuum' term, $\psi_f = -(\boldsymbol{ik}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$, which is its precise negation in both superposition and combination:

$$\psi_f + \psi_v = (\boldsymbol{ik}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m) - (\boldsymbol{ik}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m) = \mathbf{0} \tag{24}$$

$$\psi_f \psi_v = -(\boldsymbol{ik}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)(\boldsymbol{ik}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m) = \mathbf{0} \tag{25}$$

In this representation, Pauli exclusion ensures that no two fermions share the same vacuum.

Pauli exclusion is an obvious consequence of nilpotency. But, conventionally, we derive it from the fact that fermion wavefunctions are antisymmetric, so that:

$$(\psi_1\psi_2 - \psi_2\psi_1) = -(\psi_2\psi_1 - \psi_1\psi_2) \tag{26}$$

In nilpotent terms, we write $(\psi_1\psi_2 - \psi_2\psi_1)$ as

$$(\pm i\boldsymbol{k}E_1 \pm \boldsymbol{i}\mathbf{p_1} + \boldsymbol{j}m_1)(\pm i\boldsymbol{k}E_\mathbf{2} \pm \boldsymbol{i}\mathbf{p_{21}} + \boldsymbol{j}m_{\mathbf{21}}) = \mathbf{4p_1p_2} - \mathbf{4p_2p_1} = \mathbf{8}i\mathbf{p_1} \times \mathbf{p_2} = -\mathbf{8}i \times \mathbf{p_1p_2} \tag{27}$$

This result is clearly antisymmetric, but it also has a quite astonishing consequence, for it requires any nilpotent wavefunction to have a **p** vector, in *real space*, the one defined by the axes **i**, **j**, **k**, at a *different orientation* to any other. The wavefunctions of all nilpotent fermions then instantaneously correlate because the planes of their **p** vector directions must all intersect. This is the *only* source of the *entire* physical information relating to the fermion, for, at the same time, the nilpotent condition requires the iE, **p** and m combinations to be unique, and we can visualize this as constituting a unique direction in vacuum space along a set of axes defined by $\boldsymbol{k}$, $\boldsymbol{i}$, $\boldsymbol{j}$ or **k**, **i**, **j**, with coordinates defined by the values of iE, **p** and m.

Fermion singularity, in this context, can only ever result from spatial duality. The duality has to be made manifest, and specifically in a chiral manner, because the combination of the two spaces produces a chiral result. We see the duality in the characteristic spin $\frac{1}{2}$ which show fermion 'rotation' negotiating 2 spaces through the *zitterbewegung* motion and the chirality in the positive nonzero rest mass which results from this. This chirality is necessarily the same as that produced by that produced by the chirality of vacuum space in the Higgs mechanism. Berry phase in any of its manifestations (Aharonov-Bohm effect, Jahn-Teller effect, quantum Hall effect, Cooper pairing, etc.) [6] and can be seen as a consequence, or even an expression, of the singularity of the fermion state, leading to a topology with an extra twist, equivalent to spin $\frac{1}{2}$. The pole in the fermion propagator occurs at the precise 'boundary' between the two spaces, respectively characterised by positive and negative energies, (+ E) and vacuum space (- E), or forward and reverse times, (+ t) and (- t), the very combination which makes the singularity possible. The combination of two spaces becomes the same thing as the actual creation of point charges, the charges, in their extended form as sources of all gauge interactions [2], being a manifestation of the 'directions' of the vacuum space. Ultimately, through *zitterbewegung* and the Higgs mechanism, point charges are also the only source of invariant ('rest') mass, and, in the combined spaces, of relativistic quantization.

6 The nilpotent condition

So, what is the origin of this other space? Clifford algebra, significantly, has 3 subalgebras, which we can describe as scalar, complex and quaternion, or scalar, trivector and bivector. Each of these is an algebra in its own right, and it is difficult to see why only the full Clifford algebra should have a physical meaning. In fact, previous work suggests that all of the subalgebras have physical meanings on the same level as Clifford algebra, and that they represent the respective physical concepts of mass, time and charge [2,7]. That is, besides the vector algebra of space, we have three independent algebras which have a physical representation on the same level as space. If we combine these the three physical concepts as representing everything that is excluded from space as represented by **I**, **J**, **K**, then the total structure is equivalent to a single vector space represented by these units, but *without anything which directly corresponds to them*:

charge	$\boldsymbol{i}$	$\boldsymbol{j}$	$\boldsymbol{k}$	bivector	pseudovector	quaternion
time	i			trivector	pseudoscalar	
mass	1			scalar		

Of course, when we refer here to 'charge', we mean the quantity that is the source of the electric, strong and weak gauge fields *before* the symmetry-breaking which will result from the 'compactification' of the units into the minimum number of group generators, a kind of 'grand unified' value.

The space represented by $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ or **I**, **J**, **K** is then never observed directly, because it is a mathematical composite, not a physical quantity. In addition to its effect in generating the new combined quantized quantities of energy, momentum and rest mass, we can also see that the symmetry breaking between the units of the 'vacuum' space can be seen as generating the symmetry breaking that is observed in the units of weak, strong and electric charge, which are respectively associated with pseudoscalar, vector and scalar quantities, and the $SU(2)$, $SU(3)$ and $U(1)$ group symmetries which are ultimately derived from them: [2]

$i\boldsymbol{k}$	$\boldsymbol{ii}\ \boldsymbol{ij}\ \boldsymbol{ik}$	$1\boldsymbol{j}$
K	i**I**i i**I**j i**I**k	i**J**
energy	*momentum*	*mass*
weak charge	*strong charge*	*electric charge*
$SU(2)$	$SU(3)$	$U(1)$

We can even see how the Clifford algebra extends to 10 dimensions of the kind required by string theory (5 for energy, momentum and mass and 5 for charge), with 6 fixed or compactified (that is, all except energy and momentum) [2].

7 The physical significance of the H4 algebra

Besides the algebraic properties defined by their units, which may be described as real (norm 1) / imaginary (norm 1), and commutative (1D) / anticommutative (3D), the conserved and nonconserved natures of charge and space are related to the way they are combined in the 5 group generators creating the norm 0 overall structure, while the corresponding natures of mass and time are related to the fact that quantities with their algebraic characteristics are needed to complete the quaternion and vector properties of charge and space [2,7].

Work done over the last decade suggests that we should we take mass, time, charge and space as successive descriptions of the universe generated by a 'universal rewrite system', with their four commutative algebras existing as a simultaneous description [2,8,9]. The first two of these are scalar and complex, so the description reduces to a combination of scalar, complex and quaternion acting as though it were a vector space, together with another vector space. The combination, which we have called 'vacuum space', remains unobservable because it is not physical. However, the breaking of the symmetry of this 'space' which occurs when we create the 5 generators of the algebra becomes the ultimate source of the breaking of symmetry between the physical interactions.

In fact, this, along with previous work [2,7,10,12] suggests that the fundamental parameters mass, time, charge and space have a fundamental relationship (before combination) which can be identified as a Klein-4 symmetry:

mass	real (norm +1)	commutative	conserved
time	imaginary (norm -1)	commutative	nonconserved
charge	imaginary (norm -1)	anticommutative (3D))	conserved
space	real (norm +1)	anticommutative (3D)	nonconserved

This symmetry, as we have seen, is the same as that of the H4 double algebra in which the two spaces have equal status, and we can equally arrange the parameters mass (M), time (T), charge (C) and space (S) in equivalent tables reflecting this algebra, for example:

*		M	T	C	S
M		M	T	C	S
T		T	M	S	C
C		C	S	M	T
S		S	T	C	M

Similar considerations could be applied to identity and the T, P and C transformations, as these are related to the respective properties of mass, time, charge and space. Their fundamental algebraic units, which are respectively scalar, pseudoscalar, quaternion and vector, also have a Klein-4 symmetry when expressed as the Clifford algebra equivalents scalar, trivector, bivector and vector (where the last two are taken over a resultant dimension).

8 Defining a dual space spinor

Though the double quadratic nature of the parameters is manifested in these structures, we can, in fact, also relate this characteristic, as manifested in the spin property of point particles, to a quartic geometry which preserves the exact equivalence of the two spaces. One way of generating the four solutions for the wavefunction, say ψ, required by the Dirac equation is to multiply it by a 4-spinor, a summation of 4 terms which adds to 1. The four terms in the wavefunction then become ψ multiplied each of the 4 terms in the spinor, with individual terms in the spinor used as projection operators to project out individual states: fermion / antifermion and spin up / down. In the nilpotent formalism, spinors are not directly necessary because the terms are already projected, but the formalism can be set up in such a way that spinors can be used. The way which seems to be most convenient is to use both pre- and post-multiplication of ψ, as with the C, P, T operators.

A 4-spinor requires a set of primitive idempotents which add up to 1, and are orthogonal, with products between them being 0. In nilpotent quantum mechanics, we can define such idempotents in terms of the H4 algebra, constructed from the dual vector spaces:

$$\begin{gathered}(1-\mathbf{iI}-\mathbf{jJ}-\mathbf{kK})/4\\(1-\mathbf{iI}+\mathbf{jJ}+\mathbf{kK})/4\\(1+\mathbf{iI}-\mathbf{jJ}+\mathbf{kK})/4\\(1+\mathbf{iI}+\mathbf{jJ}-\mathbf{kK})/4\end{gathered}$$

Here, we see immediately that the 4 terms add up to 1, and that they are orthogonal as well as idempotent.

We can generate the same terms using coupled quaternions rather than vectors:

$$\begin{gathered}(1+\boldsymbol{iI}+\boldsymbol{jJ}+\boldsymbol{iI})/4\\(1+\boldsymbol{iI}-\boldsymbol{jJ}-\boldsymbol{iI})/4\\(1-\boldsymbol{iI}+\boldsymbol{jJ}-\boldsymbol{iI})/4\\(1-\boldsymbol{iI}-\boldsymbol{jJ}+\boldsymbol{iI})/4\end{gathered}$$

Though the 'spaces' in these structures are completely dual, since the corresponding units from the two spaces are always paired, the system is nevertheless unavoidably chiral, as the signs cannot be completely reversed. We can only reverse any two of them. For example:

$$\begin{gathered}(1+\mathbf{iI}-\mathbf{jJ}+\mathbf{kK})/4\\(1+\mathbf{iI}+\mathbf{jJ}-\mathbf{kK})/4\\(1-\mathbf{iI}-\mathbf{jJ}-\mathbf{kK})/4\\(1-\mathbf{iI}+\mathbf{jJ}+\mathbf{kK})/4\end{gathered}$$

The full 'spinor' form of the nilpotent wavefunction can be recovered by pre- and post-multiplying a 'pre-spinor' form of the nilpotent either by the original set of double vector spinors, or the set with signs reversed. A typical result would be:

$$\begin{pmatrix}1&0&0&0\\0&-i\boldsymbol{k}\mathbf{k}&0&0\\0&0&-i\boldsymbol{i}\mathbf{i}&0\\0&0&0&-i\boldsymbol{j}\mathbf{j}\end{pmatrix}\begin{pmatrix}i\boldsymbol{k}E+\boldsymbol{i}\mathbf{p}+\boldsymbol{j}m\\i\boldsymbol{k}E+\boldsymbol{i}\mathbf{p}+\boldsymbol{j}m\\i\boldsymbol{k}E+\boldsymbol{i}\mathbf{p}+\boldsymbol{j}m\\i\boldsymbol{k}E+\boldsymbol{i}\mathbf{p}+\boldsymbol{j}m\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&i\boldsymbol{k}\mathbf{k}&0&0\\0&0&i\boldsymbol{i}\mathbf{i}&0\\0&0&0&-i\boldsymbol{j}\mathbf{j}\end{pmatrix}$$

$$=\begin{pmatrix}(i\boldsymbol{k}E+\boldsymbol{i}\mathbf{p}+\boldsymbol{j}m) & (i\boldsymbol{k}E-\boldsymbol{i}\mathbf{p}+\boldsymbol{j}m) & (-i\boldsymbol{k}E+\boldsymbol{i}\mathbf{p}+\boldsymbol{j}m) & (-i\boldsymbol{k}E-\boldsymbol{i}\mathbf{p}+\boldsymbol{j}m)\end{pmatrix} \tag{28}$$

Here, the chirality is assigned to the mass term.

The spinor structures we have generated have the exact form of the components of the two forms of the Berwald-Moor metric, a structure in a quartic space:

$$(t - x - y - z)(t - x + y + z)(t + x - y + z)(t + x - y + z) \tag{29}$$

$$(t + x + y + z)(t + x - y - z)(t - x + y - z)(t - x - y + z) \tag{30}$$

The quartic Berwald-Moor metric can be seen as an expression of the fundamentally rotationally quartic nature of the underlying algebra. Multiplication of the units of the algebra produces rotations in the spaces and generates identity after a complete cycle. Multiplication of the spin metric, producing a zero product, shows that it describes a singularity. The perfect duality between the two component spaces manifested in the spinor structure (which ultimately conveys all the information in a fermionic system) means that, in principle, we could restructure physical equations to locate the singularity in real space, rather than spinor (or vacuum) space.

The generation of a quartic space from two quadratic ones is related to the fact that the spinor structure ultimately comes from 4×4 matrices which are themselves products two sets of 2×2 matrices, each of which are isomorphic to the units of the usual quadratic vector spaces. In the case of the primitive idempotent spinors, multiplying the 4 components in any order will always produce a zero product, in effect defining a singularity in 'spinor space'. This singularity is identifiable as the one produced by applying the nilpotent condition, which, as we have seen, distorts the vacuum (or spinor) space.

The significance of the H4 algebra and the Klein-4 group can be seen from their many manifestations at a fundamental level in physics. Another one may be seen in their general relevance to the creation of an information structure in any self-organizing system, of which the fermion is a classic example [12]. Many people have been interested in creating a picture of the physical world in terms of cellular automata. Here, we imagine a 3-dimensional grid of cells which are either occupied or unoccupied, with the state of occupation being determined according to a number of fixed rules. In a quantum mechanical world, however, composed of point-like particles, the cell size would have to be reduced to infinitesimal size, and there would be an infinite number of possible cells in any given space. The key fact here is that long-lived correlation in cellular automata can only be accomplished through the Klein-4 group, exactly as we have accomplished using the spin property of fermions.

9 Conclusion

While many people have thought that a redefinition of space might lead to a description of physics in terms of a single concept, a system constructed from dual vector spaces, each of which is commutative to the other, can be used immediately to construct relativistic quantum mechanics and a description of the fermion state with the required properties. Examination of the structure reveals that the two spaces contain identical information. The apparently broken symmetry between the two spaces observed through the quadratic geometry of ordinary space becomes a perfect and unbroken symmetry in the quartic geometry which defines the single physical quantity through which the two spaces can be combined. The symmetry appears broken to the observer using the quadratic geometry of ordinary space because the underlying group structure requires 5 generators, which automatically leads to a broken symmetry. However, the spin structure that connects the two spaces can be described by a quartic geometry which manifests a perfect and unbroken symmetry between the two component spaces. The Klein-4 group appears to be the symmetry which is most significant at the fundamental level, and its equivalent significance in the theory of cellular automata hints at another way in which the overall structure could be made manifest.

References

[1] Hestenes D. Space-Time Algebras. Gordon and Breach, New York, 1966.

[2] Rowlands P. Zero to Infinity: The Foundations of Physics, World Scientific, Singapore and Hackensack, NJ, 2007.

[3] Rowlands P. Physical interpretations of nilpotent quantum mechanics, arXiv: 1004.1523, 2010.

[4] Penrose R. Twistor quantization and the curvature of spacetime // *Int J Theor. Phys*, 1968, 1, 61–99.

[5] Penrose R. The Road to Reality. Jonathan Cape, London, 2004.

[6] Berry M.V. Quantal phase factors accompanying adiabatic changes // *Proc. R. Soc. Lond. A*, 1984, 392 45–57.

[7] Rowlands P. The fundamental parameters of physics // *Speculat Sci Tech*, 1983, 6, 69–80.

[8] Rowlands P., Diaz B. A universal alphabet and rewrite system, arXiv:cs.OH/0209026, 2002.

[9] Diaz B., Rowlands P. A computational path to the nilpotent Dirac equation // *Int J Comp Anticip Syst*, 2005, 16, 203–18.

[10] Rowlands P. A foundational approach to physics, arXiv:physics/0106054 (2001)

[11] Rowlands P. Are there alternatives to our present theories of physical reality?, arXiv:0912.3433, 2009.

[12] Marcer P., Rowlands P. The 'logic' of self-organizing systems, AAAI Technical Reports 2010-08-020, 2010.

ДУАЛЬНЫЕ ПРОСТРАНСТВА, СИНГУЛЯРНОСТИ ЧАСТИЦ И ГЕОМЕТРИИ ЧЕТВЕРТОГО ПОРЯДКА

Питер Роуландс

Университет Ливерпуля, Ливерпуль, ЮК

p.rowlands@liverpool.ac.uk

Релятивистская квантовая механика и свойства фермионов Дирака могут быть получены с использованием коммутирующих между собой идентичных 2-векторных пространств. Очевидное нарушение симметрии между этими двумя пространствами, наблюдаемое через геометрию обычного пространства, становится совершенно симметричным в геометрии четвертого порядка, которая определяет единичную физическую величину через которую эти два пространства могут комбинироваться.

Ключевые слова: дуальное векторное пространство, нильпотентная квантовая механика, вакуумное пространство, фермионная сингулярность, H4-алгебра.