GEOMETRIC PROPERTIES OF EINSTEIN'S LAW OF ADDITION OF VELOCITIES AND QUATERNIONIC LAW OF ADDITION

Mushfiq Ahmad

Rajshahi University, Rajshahi, Bangladesh

mushfiqahmad@gmail.com

If velocities **u** and **v** add up to give **w**. The three velocities form a triangle. The same velocities, but in the opposite direction, $-\mathbf{v}$ and $-\mathbf{u}$ should add up to give $-\mathbf{w}$. Isotropy of space requires that the reversal of direction should reverse the order of addition — $-\mathbf{v}$ should come before $-\mathbf{u}$. Lorentz Einstein addition does not fulfill this requirement and Wigner rotation in invoked to correct it. Reciprocal symmetric transformation, we are proposing, maintains the isotropy of space and Wigner rotation is not needed.

Key Words: Einstein's law, composition of velocities, associativity, non-associativity, Wigner-Thomas rotation, Thomas precession, Pauli matrices, Pauli quaternion, quaternionic addition, geometry, isotropic.

1 Introduction

Einstein's law of composition of velocities (1.1) «is neither commutative nor associative» [1].

$$\mathbf{v} \oplus \mathbf{u} = \frac{\mathbf{v} + \mathbf{u}/\lambda_v + \{1 - 1/\lambda_v\} \dfrac{\mathbf{u}.\mathbf{v}}{\mathbf{v}^2}\mathbf{v}}{1 + \mathbf{u} \cdot \mathbf{v}/c^2} \tag{1.1}$$

$$\gamma_v = \frac{1}{\sqrt{1 - (v/c)^2}}. \tag{1.2}$$

Non-associativity leads to ambiguity [2]. The sum of the 3 velocities below can be $\mathbf{u}''$ or $\ddot{\mathbf{u}}$

$$\mathbf{u}'' = \mathbf{u} \oplus (\mathbf{v} \oplus \mathbf{w}) \neq (\mathbf{u} \oplus \mathbf{v}) \oplus \mathbf{w} = \ddot{\mathbf{u}}\,. \tag{1.3}$$

To achieve an agreement between $\mathbf{u}''$ and $\ddot{\mathbf{u}}$ one includes a Wigner-Thomas [3] rotation to $\mathbf{u}''$ or $\ddot{\mathbf{u}}$. Ungar has given [1] a set of prescriptions to rotate. The ambiguity persists because one has to decide arbitrarily whether to rotate $\mathbf{u}''$ to agree with $\ddot{\mathbf{u}}$ or vice versa. To have a unique relative velocity, $\mathbf{u}''$ or $\ddot{\mathbf{u}}$ and $\mathbf{w}'$ or $-\mathbf{w}$ we have to choose a (preferred) frame of reference. Oziewicz wrote, «We must violate the Relativity Principle in order to have the unique Einstein's relative velocity» [4].

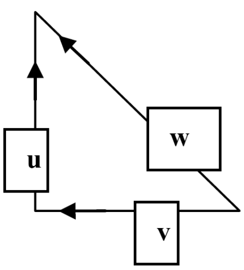

Figure 1.

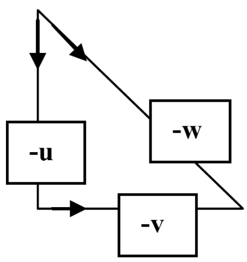

Figure 2.

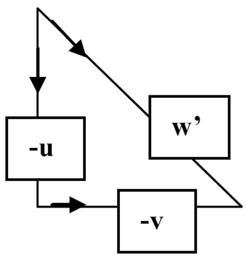

Figure 3.

Another implication of the non-associativity is Mocanu paradox [5], $\mathbf{w}' \neq -\mathbf{w}$, the inequality below: $\mathbf{w}'$ and $-\mathbf{w}$ from Fig. 3 and Fig. 1 above

$$\mathbf{w}' = (-\mathbf{u}) \oplus (-\mathbf{v}) \neq -(\mathbf{v} \oplus \mathbf{u}) = -\mathbf{w} \tag{1.4}$$

«There have been attempts [6] to explain the non-associativity, and also Mocanu paradox, as the Thomas rotation. ... We consider this attempt not satisfactory. ... Dirac in 1928 explained ... the correct spin levels in terms of the Clifford algebra and the Dirac equation, without invoking the Thomas rotation. The Dirac equation conceptually ought to be understood in terms of the Clifford algebra alone. No longer did anyone need Thomas's precession except for the non-associative $\oplus$-addition of velocities» [4].

Ahmad has proposed a Clifford algebraic [7, 6.8] reciprocal symmetric transformation

$$\mathbf{w}_Q = \mathbf{v} \oplus_Q \mathbf{u} = \frac{\mathbf{v} + \mathbf{u} + i\mathbf{v} \times \mathbf{u}/c}{\mathbf{1} + \mathbf{v} \cdot \mathbf{u}/c^2} \tag{1.5}$$

which is associative [2] and resolves [7] Mocanu paradox (1.4).

2 Anisotropy and Geometric Properties of Einstein's Law of Addition

Every one of $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ can be written (in 3 different ways) as the sum of 2 other velocities as in velocity triangle Fig. 1

$$\begin{aligned} &\text{(i)} \quad \mathbf{w} = \mathbf{v} \oplus \mathbf{u}, \\ &\text{(ii)} \quad \mathbf{u} = (-\mathbf{v}) \oplus \mathbf{w}, \\ &\text{(iii)} \quad \mathbf{v} = \mathbf{w} \oplus (-\mathbf{u}) \end{aligned} \tag{2.1}$$

Every one of $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ can be written (in 3 different ways) as the sum of 2 other velocities as in velocity triangle Fig. 1

$$\begin{aligned} &\text{(i)} \quad -\mathbf{w} = (-\mathbf{u}) \oplus (-\mathbf{v}), \\ &\text{(ii)} -\mathbf{u} = (-\mathbf{w}) \oplus \mathbf{v} \\ &\text{(iii)} \quad -\mathbf{v} = \mathbf{u} \oplus (-\mathbf{w}) \end{aligned} \tag{2.2}$$

Isotropy requires that all the six expressions be equivalent (consistent). We shall study this consistency below. We shall use (1.1) to calculate $\mathbf{w} \cdot \mathbf{u}$ from (i) of (2.1) and from (i) of (2.2) and compare them.

From Fig. 1 using (1.1) we get

$$\mathbf{w} = \mathbf{v} \oplus \mathbf{u} = \frac{\mathbf{v} + \mathbf{u}/\lambda_v + \{1 - 1/\lambda_v\} \dfrac{\mathbf{u}.\mathbf{v}}{\mathbf{v}^2}\mathbf{v}}{1 + \mathbf{u} \cdot \mathbf{v}/c^2} \tag{2.3}$$

Taking its dot product with $\mathbf{v}$

$$\mathbf{w} \cdot \mathbf{v} = \frac{\mathbf{v}^2 + \mathbf{u} \cdot \mathbf{v}/\lambda_v + \{1 - 1/\lambda_v\} \dfrac{\mathbf{u}.\mathbf{v}}{\mathbf{v}^2}\mathbf{v}^2}{1 + \mathbf{u} \cdot \mathbf{v}/c^2} \quad = \frac{\mathbf{v}^2 + \mathbf{u} \cdot \mathbf{v}}{1 + \mathbf{u} \cdot \mathbf{v}/c^2} \tag{2.4}$$

Again from Fig. 2 using (1.1) we get

$$-\mathbf{w} = (-\mathbf{u}) \oplus (-\mathbf{v}) = -\frac{\mathbf{u} + \mathbf{v}/\lambda_u + \{1 - 1/\lambda_u\} \dfrac{\mathbf{u}.\mathbf{v}}{\mathbf{u}^2}\mathbf{u}}{1 + \mathbf{u} \cdot \mathbf{v}/c^2} \tag{2.5}$$

Or

$$\mathbf{w} = \frac{\mathbf{u} + \mathbf{v}/\lambda_u + \{1 - 1/\lambda_u\} \dfrac{\mathbf{u}.\mathbf{v}}{\mathbf{u}^2}\mathbf{u}}{1 + \mathbf{u} \cdot \mathbf{v}/c^2} \tag{2.6}$$

Taking its dot product with $\mathbf{v}$

$$\mathbf{w}\cdot\mathbf{v} = \frac{\mathbf{u}\cdot\mathbf{v} + \mathbf{v}^2/\lambda_u + \{1 - 1/\lambda_u\}\dfrac{(\mathbf{u}.\mathbf{v})^2}{\mathbf{u}^2}}{1 + \mathbf{u}\cdot\mathbf{v}/c^2} \tag{2.7}$$

(2.4) and (2.7) give different results except when $\mathbf{u}$ and $\mathbf{v}$ are collinear.

Now we shall calculate $\mathbf{w}\cdot\mathbf{u}$ using (2.3) and compare with (2.7) for the case $|\mathbf{u}| = |\mathbf{v}|$. Isotropy requires that for this case $|\mathbf{w}\cdot\mathbf{v}| = |\mathbf{w}\cdot\mathbf{u}|$. We shall study this below. Taking the dot product of (2.3) we get

$$\mathbf{w}\cdot\mathbf{u} = \frac{\mathbf{v}\cdot\mathbf{u} + \mathbf{u}^2/\lambda_v + \{1 - 1/\lambda_v\}\dfrac{(\mathbf{u}.\mathbf{v})^2}{\mathbf{v}^2}}{1 + \mathbf{u}\cdot\mathbf{v}/c^2} \tag{2.8}$$

Using $|\mathbf{u}| = |\mathbf{v}|$, (2.4) gives

$$\mathbf{w}\cdot\mathbf{v} \quad = \frac{\mathbf{u}^2 + \mathbf{u}\cdot\mathbf{v}}{1 + \mathbf{u}\cdot\mathbf{v}/c^2} \tag{2.9}$$

Again (2.8) and (2.9) give different results except when $\mathbf{u}$ and $\mathbf{v}$ are collinear. In particular, if $\mathbf{u}$, and $\mathbf{v}$ are mutually orthogonal so that $\mathbf{u}\cdot\mathbf{v} = 0$, (2.8) and (2.9) give, contrary to our requirement, the inconsistent result $|\mathbf{w}\cdot\mathbf{v}| \neq |\mathbf{w}\cdot\mathbf{u}|$

$$\mathbf{w}\cdot\mathbf{u} = \mathbf{u}^2/\lambda_v \qquad \text{and} \qquad \mathbf{w}\cdot\mathbf{v} = \mathbf{u}^2. \tag{2.10}$$

We conclude, therefore, that Einstein's law (1.1) describes an anisotropic and inhomogeneous geometry.

3 Thomas Precession and General Validity

When applying (1.1) in situations involving electrons, it might be possible to get reasonable results invoking Thomas precession [1], but then, the validity of Einstein's law (1.1) will be limited to such cases and Special Relativity loses its general validity. We need a law of addition which represents a mathematically valid isotropic geometry. Quaternionic reciprocal symmetric transformation (1.5) fulfills our requirements.

4 Quaternionic Transformation and Isotropic Geometry

Following the triangles in Fig. 1 and Fig. 2, quaternionic transformation (1.5) gives the same relation (1.5). Therefore, corresponding to (2.4) and (2.7) we get the same relation in this case, and there is no inconsistency.

5 Conclusion

Einstein's law of addition of velocities does not represent the geometry of an isotropic space. In recognition of this lack of mathematical validity, the is sometimes called Einstein's law of «composition» [8] [instead of «addition»] of velocities.

In the application of the law in situations involving electrons Thomas precession provides a correction, but Dirac theory puts the validity of this correction in doubt. Quaternionic transformation gives a law of addition of velocities fulfilling Einstein's requirements and provides a mathematically valid representation of the geometry of an isotropic space; and Thomas precession correction is not needed.

6 Appendix

We write using (1.2)

$$\hat{u}_{\pm} = \gamma_u \{\sigma_0 c \pm (\sigma_x u_x + \sigma_y u_y + \sigma_z u_z)\} \tag{6.1}$$

And require

$$\hat{u}_+ \hat{u}_- = \frac{1}{1-(u/c)^2}\{c^2 - u_x^2 - u_y^2 - u_z^2\} = c^2 \tag{6.2}$$

Consistency between (6.1) and (6.2) will be guaranteed if [9]

$$\sigma_i \sigma_j + \sigma_j \sigma_i = 0 \quad \text{for} \quad i \neq j \tag{6.3}$$

$$\sigma_i \sigma_0 = \sigma_0 \sigma_i \tag{6.4}$$

$$\sigma_i^2 = 1 \quad \text{and} \quad \sigma_0^2 = 1\,. \tag{6.5}$$

To complete we also define [10]

$$\sigma_x \sigma_y = i\sigma_z . \varepsilon_{xyz} \tag{6.6}$$

Now we can form the product

$$\hat{u}_+ \left(\hat{v}_-/c\right) = \gamma_u \gamma_v \{\sigma_0 c + (\sigma_x u_x + \sigma_y u_y + \sigma_z u_z)\} \cdot \{\sigma_0 c - (\sigma_x v_x + \sigma_y v_y + \sigma_z v_z)\} \tag{6.7}$$

Using (6.3) – (6.6) we get

$$\begin{aligned} \hat{u}_+(\hat{v}_-/c) &= (\gamma_u \gamma_v/c).\,(c^2 - \mathbf{u}\cdot\mathbf{v} + c.\sigma\cdot\ (\mathbf{u}-\mathbf{v}) - i\sigma\cdot(\mathbf{u}\times\mathbf{v})) \\ &= \gamma_y \left\{c + \sigma\cdot\frac{(\mathbf{u}-\mathbf{v}) - i\,(\mathbf{u}\times\mathbf{v})\,/c}{1-\mathbf{u}\cdot\mathbf{v}/c^2}\right\} = y_y\,\{c + \sigma\cdot\mathbf{y}\} \end{aligned} \tag{6.8}$$

Where

$$\gamma_y = \gamma_u.\gamma_v.\left(1 - (\mathbf{u}\cdot\mathbf{v})\,/c^2\right). \tag{6.9}$$

The σ's permit [11] the following matrix representations

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{and} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{6.10}$$

References

[1] Ungar A.A. Beyond the Einstein Addition Law and its Gyroscopic Thomas Precession. Kluwer Academic Publishers. New York, Moscow, 2001, 464 p.

[2] Ahmad M., Alam M.S. Relativistic requirement and comparison between reciprocal symmetric transformation and Lorentz transformation // *Physics Essays*, 22(2), Vol. 22, 2009, pp. 164 – 167.

[3] Wigner E.P. On the unitary representations of the inhomogeneous Lorentz group // *Ann. Math.* 40, 1939, pp. 149 – 204.

[4] Zbigniew Oziewicz. Physical Interpretations of Relativity Theory. Moscow 2007, pp. 292 – 303.

[5] Mocanu C. On the relativistic velocity composition paradox and the Thomas rotation // *Foundation of Physics Letters*, 5(5), Vol. 5, 1992, pp. 443 – 456.

[6] Ungar A.A. The Thomas Formalism Underlying A Non-Associative Group Structure for Relativistic Velocities // *Applied Mathematics Letters*, 1(4), 1988, pp. 403 – 405.

[7] Ahmad M. Mocanu's Paradox and Quaternionic Transformation as the Answer // *Hypercomplex Numbers in Geometry and Physics*, 1 (17), Vol. 9, 2012, pp. 60 – 64.

[8] Marcel Froissart. Private communication

[9] Adler S.L. Quaternionic Quantum Mechanics and Quantum Fields. International Series of Monographs on Physics – 88.

[10] Rastall P. Quaternions in Relativity // *Reviews of Modern Physics*, 36 (3), Vol. 36, 1964, pp. 820 – 832.

[11] Schiff L.I. Quantum Mechanics. McGraw Hill New York 1968, 554 p.

ГЕОМЕТРИЧЕСКИЕ СВОЙСТВА ЭЙНШТЕЙНОВСКОГО ЗАКОНА СЛОЖЕНИЯ СКОРОСТЕЙ И ЕГО КВАТЕРНИОННЫЙ АНАЛОГ

Мушфик Ахмед

Раджшахийский университет, Раджшахи, Бангладеш

mushfiqahmad@gmail.com

Если сложить скорости **u** и **v** – получим скорость **w**. Те же скорости, но с противоположным знаком: −**u** и −**v** должны дать −**w**. Изотропия пространства требует, чтобы инверсия направления приводила к изменению порядка сложения: −**v** должно идти перед −**u**. Лоренцево сложение не удовлетворяет этому требованию и вводится вращение Вигнера, чтобы его скорректировать. Предлагаемое нами взаимно-симметричное преобразование сохраняет изотропию пространства, и вращение Вигнера не требуется.

Ключевые слова: закон Эйнштейна, сложение скоростей, ассоциативность, неассоциативность, вращение Вигнера-Томаса, прецессия Томаса, матрицы Паули, кватернионы Паули, кватернионное сложение, геометрия, изотропный.