MOCANU'S PARADOX AND QUATERNIONIC TRANSFORMATION AS THE ANSWER

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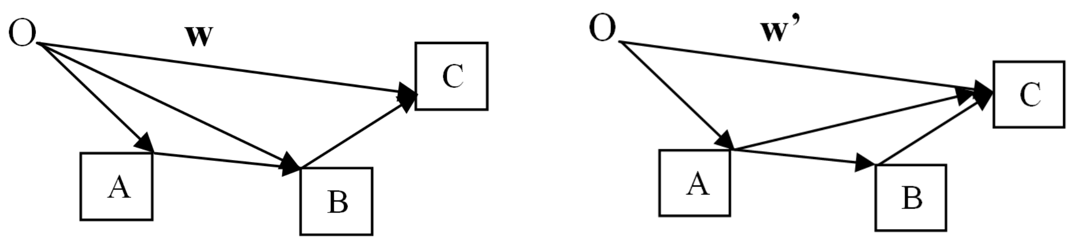
When two non colinear velocities are added following Lorentz transformation, a Wigner rotation comes into play, and reciprocity requirement is not fulfilled without this rotation: the velocity of B from A is not the negative of the velocity of A from B. Both Mocanu and Ungar have attributed the paradox (failure of reciprocity principle) to both non-commutativity and non-associativity of Einstein' law of addition of velocities. To resolve this problem, Ungar has proposed a "Weak Associative Law" (a set of corrections) to make Einstein' law of addition commutative and associative. We have shown that the paradox can be resolved without requiring commutativity. We are proposing a hypercomplex Pauli quaternion law of composition of relativistic velocities, which fulfills physical requirements. The proposed hypercomplex law compares well with Einstein's law of addition of velocities and fulfills all relativistic requirements.

Key Words: Einstein's Law, composition of velocities, associativity, non-associativity, weak associative, gyro-associative, Thomas precession, Pauli matrices, Pauli quaternion, quaternionic composition.

1 Introduction

Einstein's law of composition of velocities «is neither commutative nor associative». Ungar, however, finds it «"repairable" by gyroautomorphisms called Thomas gyrations» [1]. The process of repairing involves a series of arbitrary prescriptions and one finally achieves (in case of 3 velocities) 2 laws of addition; *right weak associative law* and *left weak associative law.* In case of more initial velocities, one will have to choose from a greater number of distinct final velocities. If A moves with respect to O (with velocity **a**), B moves with respect to A (with velocity **b**) and C with respect to B (with velocity **c**), then C moves with respect to O with a unique velocity.



In the diagrams above, according to Einstein's law of composition, $\mathbf{w} \neq \mathbf{w}'$, corresponding to [we have adopted Ungar's symbol $\oplus$ to represent Einstein's addition]

$$(\mathbf{a} \oplus \mathbf{b}) \oplus \mathbf{c} \neq \mathbf{a} \oplus (\mathbf{b} \oplus \mathbf{c}) \qquad (1.1)$$

where

$$\mathbf{w}_L = \mathbf{v} \oplus \mathbf{u} = \frac{\mathbf{v} + \mathbf{u}/\lambda_v + \{1 - 1/\lambda_v\} \dfrac{\mathbf{u}.\mathbf{v}}{\mathbf{v}^2}\mathbf{v}}{1 + \mathbf{u}\cdot\mathbf{v}/c^2} \qquad (1.2)$$

Or in a slightly different form

$$\mathbf{w}_L = \mathbf{v} \oplus \mathbf{u} = \frac{\mathbf{v} + \mathbf{u}}{1 + \mathbf{v} \cdot \mathbf{u}/c^2} + \frac{1}{c^2} \cdot \frac{\gamma_v}{\gamma_v + 1} \cdot \frac{\mathbf{v} \times (\mathbf{v} \times \mathbf{u})}{1 + \mathbf{u} \cdot \mathbf{v}/c^2} \tag{1.3}$$

$$\gamma_v = \frac{1}{\sqrt{1 - (v/c)^2}} \tag{1.4}$$

Thomas gyration prescriptions tell us how to twist $\mathbf{w}$ to make it coincide with $\mathbf{w}'$ or, alternatively, how to twist $\mathbf{w}'$ to make it coincide with $\mathbf{w}$. We have nothing to guide us which of the two velocities ($\mathbf{w}$ or $\mathbf{w}'$) we should start with.

Relativistic addition of velocities (in 3 dimensions) brings in Wignrt-Thomas rotation, without which the relativistic addition of velocities becomes frames dependent, as shown [2] below in eq. (1.6)

Using (1.2)

$$\mathbf{v} \oplus \mathbf{u} \neq -\{(-\mathbf{u}) \oplus (-\mathbf{v})\} \tag{1.5}$$

Inequality (1.5) is called Mocanu paradox. Both Mocanu [3] and Ungar [4] have attributed Mocanu's paradox (failure of reciprocity principle) to both non-commutativity and non-associativity of Einstein' law of addition of velocities. Ungar's gyrovector theory proposes corrections to make the law of addition both commutative and associative. We shall show below that commutativity is not a requirement.

Observer O_1 observes a body A moving with velocity $\mathbf{u}$ and a body B moving with velocity $\mathbf{v}$. According to O_1 the relative velocity is $\mathbf{u} \oplus (-\mathbf{v})$. A second observer O_2 is moving with velocity $\mathbf{y}$ with respect to O_1. Therefore, according to O_2 the velocities are $\mathbf{u}' = \mathbf{u} \oplus (-\mathbf{y})$ and $\mathbf{v}' = \mathbf{v} \oplus (-\mathbf{y})$ and the relative velocity is $\mathbf{u}' \oplus (-\mathbf{v}')$

$$\mathbf{u}' \oplus (-\mathbf{v}') = \{\mathbf{u} \oplus (-\mathbf{y})\} \oplus -\{\mathbf{v} \oplus (-\mathbf{y})\} \neq \mathbf{u} \oplus (-\mathbf{v}) \tag{1.6}$$

Wigner rotation was needed to explain electron spin. Dirac's theory explains electron spin without invoking Wigner rotation. Therefore, it is desirable to have an associative law of addition (without Wigner rotation) os that we can replace inequalities in (1.1) and (1.6) by corresponding equalities. We require

$$(\mathbf{a} \oplus_Q \mathbf{b}) \oplus_Q \mathbf{c} = \mathbf{a} \oplus_Q (\mathbf{b} \oplus_Q \mathbf{c}) \tag{1.7}$$

where $\oplus_Q$ (Q for quaternion) represents an associative law of addition.

2 Associative Addition and Pauli Quaternion

We consider relativistic velocities which fulfill Einstein's condition

$$|\mathbf{u}| \leq c \tag{2.1}$$

For convenience we form functions

$$\lambda_u \{1 + \mathbf{u}/c\} \tag{2.2}$$

Product $\times_L$ of these functions gives Einstein composition of velocities

$$\lambda_u \{1 + \mathbf{u}/c\} \times_L \lambda_v \{1 + \mathbf{v}/c\} = \lambda_w \{1 + (\mathbf{u} \oplus \mathbf{v})/c\} \tag{2.3}$$

Corresponding to (2.3) we shall look an for (2.4) below where $\times_L$ of (2.3) has been replaced by $\times_Q$ (Q for quaternion)

$$\lambda_u \{1 + \mathbf{u}/c\} \times_Q \lambda_v \{1 + \mathbf{v}/c\} = \lambda_w \{1 + \mathbf{w}/c\} \tag{2.4}$$

Where $\mathbf{w}$ is the associative relativistic sum of velocities $\mathbf{u}$ and $\mathbf{v}$(without Wigner rotation). We have to find $\times_Q$.

Postulate: We postulate that the $0+3$ [scalar+Cartesian [5]] vectors $(a+\mathbf{u})$ etc. of (2.2) are Pauli Quaternion [6] 4-vectors.

$$(a+\mathbf{u}) \Rightarrow (a\sigma_0 + \mathbf{u}\cdot\sigma) = a\sigma_0 + u_x\sigma_x + u_y\sigma_y + u_z\sigma_z \tag{2.5}$$

Where σ's have the following properties

$$-\sigma_y\sigma_x = \sigma_x\sigma_y = i\sigma_z \quad \text{with cyclic permutations} \tag{2.6}$$

$$\sigma_0\sigma_x = \sigma_x\sigma_0 = \sigma_x \;\text{ and }\; \sigma_x\sigma_x = 1 \text{ and also for } y \text{ and } z \text{ and } 0 \tag{2.7}$$

We replace (2.4) by

$$\lambda_u\left\{\sigma_0 + \mathbf{u}\cdot\sigma/c\right\}\lambda_v\left\{\sigma_0 + \mathbf{v}\cdot\sigma/c\right\} = \lambda_w\left\{\sigma_0 + \mathbf{w}\cdot\sigma/c\right\} \tag{2.8}$$

Using (2.6) $--$ (2.7) we have

$$\lambda_w\left\{\sigma_0 + \mathbf{w}\cdot\sigma/c\right\} = \lambda_u\lambda_v\left(1+\mathbf{u}\cdot\mathbf{v}/c^2\right)\left\{\sigma_0 + \left(\frac{\mathbf{u}+\mathbf{v}+i\mathbf{u}\times\mathbf{v}/c}{1+\mathbf{u}\cdot\mathbf{v}/c^2}\right)\cdot\sigma/c\right\} \tag{2.9}$$

Comparison between (2.8) and (2.9) gives, using $\oplus_Q$ of (1.7)

$$\mathbf{w} = \mathbf{u}\oplus_Q\mathbf{v} = \frac{\mathbf{u}+\mathbf{v}+i\mathbf{u}\times\mathbf{v}/c}{\mathbf{1}+\mathbf{u}\cdot\mathbf{v}/c^2} \tag{2.10}$$

The σ's permit the following matrix representations

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \text{ and } \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.11}$$

Using (2.10) we can replace the inequality (1.5) by the equality below

$$\mathbf{v}\oplus_Q\mathbf{u} = -\left\{(-\mathbf{u})\oplus_Q(-\mathbf{v})\right\} \tag{2.12}$$

3 Comparison between Einstein's Law and Quaternionic Law

They have the same magnitude

$$|\mathbf{u}\oplus\mathbf{v}| = |\mathbf{u}\oplus_Q\mathbf{v}| \tag{3.1}$$

The real part of $\mathbf{u}\oplus_Q\mathbf{v}$ and whole of $\mathbf{u}\oplus\mathbf{v}$ are confined to the plane of $\mathbf{u}$ and $\mathbf{v}$. Both the sums are rotated with respect to the direction of the Euclidian sum $\mathbf{u}+\mathbf{v}$. The imaginary component of $\mathbf{u}\oplus_Q\mathbf{v}$ is responsible for the rotation. The axis of rotation is orthogonal to the plane of $\mathbf{u}$ and $\mathbf{v}$.

In the limit $c\to\infty$ both the sums go to the corresponding Galilean sum $\mathbf{u}+\mathbf{v}$. We want to see below how they rotate in the other (non physical) extreme $c\to 0$. In $\mathbf{u}\oplus\mathbf{v}$ the dominating term is $\frac{1}{c^2}\cdot\frac{\gamma_u}{\gamma_u+1}\cdot\frac{\mathbf{u}\times(\mathbf{u}\times\mathbf{v})}{1+\mathbf{u}\cdot\mathbf{v}/c^2}$

$$\begin{gathered}\mathbf{u}\oplus\mathbf{v}\to\frac{1}{c^2}\cdot\frac{\gamma_u}{\gamma_u+1}\cdot\frac{\mathbf{u}\times(\mathbf{u}\times\mathbf{v})}{1+\mathbf{u}\cdot\mathbf{v}/c^2}\to\frac{1}{c^2}\frac{\sqrt{1+(\pm iu/c)^2}}{\sqrt{1-(u/c)^2}+1-(u/c)^2}\frac{\mathbf{u}\times(\mathbf{u}\times\mathbf{v})}{\mathbf{u}\cdot\mathbf{v}/c^2}\to \\ -(c/u)^2(\pm iu/c)\frac{\mathbf{u}\times(\mathbf{u}\times\mathbf{v})}{\mathbf{u}\cdot\mathbf{v}}\to\mp ic\frac{Sin\theta}{Cos\theta}\to\mp ic.\tan\theta.\mathbf{m}\end{gathered} \tag{3.2}$$

Where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$ and $\mathbf{m}$ is a unit vector in the plane of $\mathbf{u}$ and $\mathbf{v}$ and orthogonal to $\mathbf{u}$. Similarly

$$\mathbf{u} \oplus_Q \mathbf{v} \to \frac{i\mathbf{u} \times \mathbf{v}/c}{\mathbf{u} \cdot \mathbf{v}/c^2} \to ic \tan\theta \mathbf{n} \tag{3.3}$$

Where $\mathbf{n}$ is a unit vector orthogonal to the plane of $\mathbf{u}$ and $\mathbf{v}$. Therefore, in this end both the rotations are imaginary, have the same magnitude but are in mutually orthogonal directions.

From (3.1)

$$|\mathbf{u} \oplus_Q \mathbf{v}| \le c \quad \text{if} \quad |\mathbf{u}| \le c, \quad |\mathbf{v}| \le c \tag{3.4}$$

Therefore, quaternionic sum (2.10) fulfills Einstein's postulate.

4 Conclusion

Using quaternionic transformation (2.10), we have been able to resolve the paradox (1.5), and reciprocity requirement is fulfilled (2.12). Quaternionic transformation, we are proposing, is relativistic and fulfills Einstein' postulate as shown by (3.4).

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ПАРАДОКС МОКАНУ И КВАТЕРНИОННОЕ ПРЕОБРАЗОВАНИЕ КАК ОТВЕТ

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Когда две неколлинеарные скорости складываются согласно преобразованию Лоренца появляется вращение Вигнера без которого требование взаимности не выполняется: скорость от B к A не есть скорость от A к B с обратным знаком. Мокану и Унгар связали этот парадокс (нарушение принципа взаимности) с некоммутативностью и неассоциативностью энштейновского закона сложения скоростей. Чтобы решить эту проблему Унгар предложил «слабый закон ассоциативности» (набор поправок), делающий энштейновский закон сложения коммутативным и ассоциативным. В настоящей работе мы показали, что этот парадокс может быть разрешен без требования коммутативности. Нами предложен гиперкомплексный кватернионный закон сложения относительных скоростей Паули, который отвечает всем физическим требованиям. Предложенный гиперкомплексный закон находится в хорошем соответствии с законом сложения скоростей Эйнштейна и удовлетворяет всем релятивистским требованиям.

Ключевые слова: закон Эйнштейна, сложение скоростей, ассоциативность, неассоциативность, слабая ассоциативность, гироассоциативность, прецессия Томаса, матрицы Паули, кватернион Паули, кватернионное сложение.