JET RIEMANN-HAMILTON GEOMETRIZATION FOR THE CONFORMAL DEFORMED BERWALD-MOÓR QUARTIC METRIC DEPENDING ON MOMENTA

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In this paper we expose on the dual 1-jet space $J^{1*}(\mathbb{R}, M^4)$ the distinguished (d-) Riemannian geometry (in the sense of d-connection, d-torsions, d-curvatures and some gravitational-like and electromagnetic-like geometrical models) for the (t, x)-conformal deformed Berwald-Moór Hamiltonian metric of order four.

Key Words: (t, x)-conformal deformation of quartic Berwald-Moór Hamiltonian metric, canonical nonlinear connection, Cartan canonical linear connection, d-torsions and d-curvatures, geometrical Einstein-like equations.

1 Introduction

The extension of classical Riemannian geometry of physical fields (gravitational and electromagnetic) to metrics depending on directions $g_{ij}(x, y)$ or to metrics depending on momenta $g^{ij}(x, p)$ has been and still is of great interest for scientists. In our days, such possible geometric-physical abstract extensions were created by Miron, Atanasiu and their collaborators in the works [6, 7] and [3], by using geometrical ideas and distinguished methods from:

- *Finslerian framework* (where $g_{ij}(x, y)$ is 0-positive y-homogenous, and it is provided by a 1-positive y-homogenous Finslerian metric);
- *Lagrangian framework* (where $g_{ij}(x, y)$ is not necessarily 0-positive y-homogenous, but it is provided by a Lagrangian $L : TM \to \mathbb{R}$);
- *Hamiltonian framework* (where $g^{ij}(x, p)$ is not necessarily 0-positive p-homogenous, but it is provided by a Hamiltonian $H : T^*M \to \mathbb{R}$).

The Miron's geometrical construction (which is given on tangent or cotangent bundles) relies on the using of the adapted basis of horizontal vector fields $\delta/\delta x^i$, instead of the natural basis $\partial/\partial x^i$. This is because the local transformation rules of the horizontal vector fields $\delta/\delta x^i$ are tensorial ones, so more simple than the transformation rules of the vector fields $\partial/\partial x^i$ on tangent or cotangent bundles. For such a reason, all geometrical objects used in the corresponding abstract physical theory are locally described in adapted bases. From a physical point of view, the use of distinguished vector fields $\delta/\delta x^i$ is useful in order to be more simple to obey the Einstein's relativistic law that says that the form of all physical laws have to be the same in any reference system. The geometrical translation of this physical law is that any geometric-physical object used in theory must have a global geometrical character, that is it must have the same local form in any local chart of coordinates.

In Miron's physical field geometrical theory, the classical Riemannian Levi-Civita connection is replaced by Cartan canonical connection. It is well-known that in Finsler-Lagrange-Hamilton geometrical framework there are a lot of important linear distinguished (d-) connections (Cartan, Berwald, Chern-Rund or Hashiguchi, for instance). However, the use of the Cartan canonical connection is prefered because it is the single linear d-connection which is a *metrical connection* like the Levi-Civita connection in the classical Riemannian framework. An important difference between these two connections is that the Cartan connection is only a partial torsion-free connection because the Poisson brackets of the distinguished vector fields $\delta/\delta x^i$ are not generally equal to zero.

Consequently, in both gravitational and electromagnetic Finsler-Lagrange geometrical theory, Miron and his coworkers postulate:

1. The Einstein-like equations that govern the anisotropic gravitational potentials $g_{ij}(x,y)$ are the geometrical Einstein equations associated to the Cartan canonical connection, locally described in adapted bases. Note that for the particular Lagrangian of classical electrodynamics (see [6]) these Einstein-like equations become the classical Einstein equations.

2. The anisotropic electromagnetic field is defined in similar way with Riemannian one, but by replacing the isotropic Riemannian electromagnetic potentials $A_i(x)$ with the anisotropic electromagnetic potentials $y_i = g_{ij}y^j$. Moreover, the authors substitute the Levi-Civita connection with the metrical Cartan canonical connection:

$$F_{ij}(x) = A_{i;j} - A_{j;i} \longleftrightarrow F_{ij}(x,y) = y_{i|j} - y_{j|i} =$$
$$= g_{im}\left(y^m_{|j}\right) - g_{jm}\left(y^m_{|i}\right) = g_{im}\left(\frac{\delta y^m}{\delta x^j} + y^r L^m_{rj}\right) - g_{jm}\left(\frac{\delta y^m}{\delta x^i} + y^r L^m_{ri}\right) =$$
$$= g_{im}\left(\frac{\partial y^m}{\partial x^j} - N^s_j\frac{\partial y^m}{\partial y^s} + y^r L^m_{rj}\right) - g_{jm}\left(\frac{\partial y^m}{\partial x^i} - N^s_i\frac{\partial y^m}{\partial y^s} + y^r L^m_{ri}\right) =$$
$$= g_{im}\left(-N^s_j\delta^m_s + y^r L^m_{rj}\right) - g_{jm}\left(-N^s_i\delta^m_s + y^r L^m_{ri}\right),$$

where ";" is the local covariant derivative induced by Levi-Civita connection, "$_|$" is the horizontal covariant derivative induced by metrical Cartan canonical connection, N^j_i are the components of the canonical nonlinear connection (this is induced by the Euler-Lagrange equations of the given Lagrangian, and it is in the Finsler case exactly the classical Cartan nonlinear connection), and L^k_{ij} are the horizontal components of the Cartan canonical connection. It is important to note that the anisotropic electromagnetic components $F_{ij}(x,y)$ are governed by some natural Maxwell-like equations. The naturalness of these Maxwell-like equations is coming again from the particular physical case of classical electrodynamics (see [6], [7]) in which these Maxwell-like equations reduce exactly to the classical Maxwell equations. More accurately, in this particular physical situation, the Lagrangian function (that governs the movement law of a particle of mass $m \neq 0$ and electric charge e, which is displaced concomitantly into an environment endowed both with a gravitational field and an electromagnetic one) is given by

$$L(x^k, y^k) = mc\varphi_{ij}(x^k)y^iy^j + \frac{2e}{m}A_i(x^k)y^i + \mathcal{F}(x^k), \tag{1.1}$$

where the semi-Riemannian metric $\varphi_{ij}(x)$ represents the *gravitational potential* of the space of events M, $A_i(x)$ are the components of a d-tensor on the tangent bundle representing the *electromagnetic potential*, and $\mathcal{F}(x)$ is a smooth *potential function* on the manifold M. The corresponding **electromagnetic components** are given by

$$F_{ij} = -\frac{e}{2m}\left(\frac{\partial A_i}{\partial x^j} - \frac{\partial A_j}{\partial x^i}\right),$$

and the **Maxwell geometrical equations** reduce to the classical ones:

$$\sum_{\{i,j,k\}} F_{ij|k} = 0,$$

where

$$F_{ij|k} = \frac{\partial F_{ij}}{\partial x^k} - F_{mj}\gamma^m_{ik} - F_{im}\gamma^m_{jk}.$$

Finally, in what concern the Hamiltonian geometrical approach of physical fields depending on momenta, we would like to underline only the fact that the Hamiltonian theory uses the same simmetry, methods and ideas with the Lagrangian theory, via the Legendre duality induced by the classical natural Legendre transformations.

The geometric-physical Lagrangian or Hamiltonian Berwald-Moór structure [4, 8] was intensively investigated by P.K. Rashevski [14] and further fundamented and developed by D.G. Pavlov, G.I. Garas'ko and S.S. Kokarev [5, 12, 13]. In their works, the preceding Russian scientists emphasize the importance in the theory of space-time structure, gravitation and electromagnetism of the geometry produced by the classical Berwald-Moór Lagrangian metric

$$F : TM \to \mathbb{R}, \qquad F(y) = \sqrt[n]{y^1 y^2 ... y^n}, \qquad n \geq 2,$$

or by the corresponding Berwald-Moór Hamiltonian metric

$$H : T^*M \to \mathbb{R}, \qquad H(p) = \sqrt[n]{p_1 p_2 ... p_n}.$$

In such a perspective, according to the recent geometric-physical ideas proposed by Garas'ko [5], we consider that a distinguished Riemannian geometry (in the sense of d-connection, d-torsions, d-curvatures and some gravitational-like and electromagnetic-like geometrical models) for the conformal deformations of the jet Berwald-Moór Hamiltonian metric of order four is required. Note that a similar geometric-physical study for the (t, x)-conformal deformations of the jet Berwald-Moór Lagrangian metric of order four is now completely developed in the paper [9]. Also, few elements of distinguished Hamiltonian geometry produced by the cotangent quartic Berwald-Moór metric depending of momenta are already presented in the paper [1].

In such a geometrical and physical context, this paper investigates on the dual 1-jet space $J^{1*}(\mathbb{R}, M^4)$ the Riemann-Hamilton distinguished geometry (together with a theoretical-geometric field-like theory) for the (t, x)-*conformal deformed Berwald-Moór Hamiltonian metric of order four*[1]

$$H(t, x, p) = 2e^{-\sigma(x)} \sqrt{h_{11}(t)} \left[p_1^1 p_2^1 p_3^1 p_4^1\right]^{1/4}, \tag{1.2}$$

where $\sigma(x)$ is a smooth non-constant function on M^4, $h_{11}(t)$ is a Riemannian metric on $\mathbb{R}$, and $(t, x, p) = (t, x^1, x^2, x^3, x^4, p_1^1, p_2^1, p_3^1, p_4^1)$ are the coordinates of the momentum phase space $J^{1*}(\mathbb{R}, M^4)$; these transform by the rules (the Einstein convention of summation is assumed everywhere):

$$\widetilde{t} = \widetilde{t}(t), \quad \widetilde{x}^i = \widetilde{x}^i(x^j), \quad \widetilde{p}_i^1 = \frac{\partial x^j}{\partial \widetilde{x}^i} \frac{d\widetilde{t}}{dt} p_j^1, \tag{1.3}$$

where $i, j = \overline{1, 4}$, rank $(\partial \widetilde{x}^i / \partial x^j) = 4$ and $d\widetilde{t}/dt \neq 0$. It is important to note that, based on the geometrical ideas promoted by Miron, Hrimiuc, Shimada and Sabău in the classical Hamiltonian geometry of cotangent bundles ([7]), together with those used by Atanasiu, Neagu and Oană in the geometry of dual 1-jet spaces, the differential geometry (in the sense of d-connections, d-torsions, d-curvatures, abstract gravitational-like and electromagnetic-like geometrical theories) produced by a jet Hamiltonian function $H : J^{1*}(\mathbb{R}, M^n) \to \mathbb{R}$ is now completely done in the papers [2,3,10] and [11]. In what follows, we apply the general geometrical results from [10] and [11] to the square of Hamiltonian metric (1.2), which is given by $(n = 4)$

$$\overset{*}{H}(t, x, p) = H^2(t, x, p) = 4e^{-2\sigma(x)} h_{11}(t) \left[p_1^1 p_2^1 p_3^1 p_4^1\right]^{1/2}. \tag{1.4}$$

[1]We assume that we have $p_1^1 p_2^1 p_3^1 p_4^1 > 0$. This is one domain where we can p-differentiate the Hamiltonian function $H(t, x, p)$.

Proposition 1.1 The momentum Hamiltonian metric (1.4) is exactly the natural Hamiltonian attached to the square of the jet Berwald-Moór Lagrangian metric, which has the expression

$$\overset{*}{L}(t,x,y) = e^{2\sigma(x)} h^{11}(t) \left[y_1^1 y_1^2 y_1^3 y_1^4 \right]^{1/2}. \tag{1.5}$$

Proof. The Hamiltonian $\overset{*}{H}$, associated to the square of the jet Berwald-Moór Lagrangian metric $\overset{*}{L}$, is defined by formula

$$\overset{*}{H}(t,x,p) = \sum_{r=1}^{4} p_r^1 y_1^r - \overset{*}{L}(t,x,y),$$

where

$$p_r^1 = \frac{\partial \overset{*}{L}}{\partial y_1^r} = \frac{e^{2\sigma(x)} h^{11}(t)}{2} \cdot \frac{\left[y_1^1 y_1^2 y_1^3 y_1^4 \right]^{1/2}}{y_1^r} \Leftrightarrow p_r^1 y_1^r = \frac{\overset{*}{L}}{2} \text{ (no sum by } r\text{)}. \tag{1.6}$$

It follows that we have

$$\overset{*}{H} = \sum_{r=1}^{4} p_r^1 y_1^r - \overset{*}{L} = 2\overset{*}{L} - \overset{*}{L} = \overset{*}{L}.$$

But, the equalities (1.6) obviously imply

$$p_1^1 p_2^1 p_3^1 p_4^1 = \frac{\left[\overset{*}{L} \right]^4}{16 \left[y_1^1 y_1^2 y_1^3 y_1^4 \right]} = \frac{e^{8\sigma(x)} \left[h^{11}(t) \right]^4 \left[y_1^1 y_1^2 y_1^3 y_1^4 \right]}{16} = \frac{e^{4\sigma(x)} \left[h^{11}(t) \right]^2 \left[\overset{*}{L} \right]^2}{16}.$$

In conclusion, we obtained what we were looking for:

$$\overset{*}{H} = \overset{*}{L} = 4 e^{-2\sigma(x)} h_{11}(t) \left[p_1^1 p_2^1 p_3^1 p_4^1 \right]^{1/2}.$$

Remark 1.2 Note that the jet Lagrangian metric (1.5) is even the square of the conformal deformed jet quartic Berwald-Moór Finslerian metric

$$F(t,x,y) = e^{\sigma(x)} \sqrt{h^{11}(t)} \left[y_1^1 y_1^2 y_1^3 y_1^4 \right]^{1/4}.$$

In other words, we have $\overset{*}{L} = F^2$.

2 The canonical nonlinear connection

Using the notation $\mathcal{P}^{1111} := p_1^1 p_2^1 p_3^1 p_4^1$ and taking into account that we have

$$\frac{\partial \mathcal{P}^{1111}}{\partial p_i^1} = \frac{\mathcal{P}^{1111}}{p_i^1},$$

then the *fundamental metrical d-tensor* produced by the metric (1.4) is given by the formula (no sum by i or j)

$$\overset{*}{g}{}^{ij}(t,x,p) = \frac{h^{11}(t)}{2} \frac{\partial^2 \overset{*}{H}}{\partial p_i^1 \partial p_j^1} = \frac{e^{-2\sigma(x)} \left(1 - 2\delta^{ij} \right) \left[\mathcal{P}^{1111} \right]^{1/2}}{2} \frac{}{p_i^1 p_j^1}. \tag{2.1}$$

Moreover, the matrix $\overset{*}{g} = (\overset{*}{g}{}^{ij})$ admits the inverse $\overset{*}{g}{}^{-1} = (\overset{*}{g}_{jk})$, whose entries are given by

$$\overset{*}{g}_{jk} = \frac{e^{2\sigma(x)}\left(1-2\delta_{jk}\right)\left[\mathcal{P}^{1111}\right]^{-1/2}}{2} p_j^1 p_k^1 \text{ (no sum by } j \text{ or } k). \tag{2.2}$$

Let us consider the Christoffel symbol of the Riemannian metric $h_{11}(t)$ on $\mathbb{R}$, which is given by

$$\mathrm{K}_{11}^1 = \frac{h^{11}}{2}\frac{dh_{11}}{dt},$$

where $h^{11} = 1/h_{11} > 0$. Then, using the notation $\sigma_i := \partial\sigma/\partial x^i$, we find the following geometrical result:

Proposition 2.1 For the (t,x)-conformal deformed Berwald-Moór Hamiltonian metric of order four (1.4), the **canonical nonlinear connection** on the dual 1-jet space $J^{1*}(\mathbb{R}, M^4)$ has the components

$$N = \left(\underset{1}{N}{}_{(i)1}^{(1)} = \mathrm{K}_{11}^1 p_i^1,\ \underset{2}{N}{}_{(i)j}^{(1)} = -4\sigma_i p_i^1 \delta_{ij}\right). \tag{2.3}$$

Proof. The canonical nonlinear connection produced by $\overset{*}{H}$ on the dual 1-jet space $J^{1*}(\mathbb{R}, M^4)$ has the following components (see [2]):

$$\underset{1}{N}{}_{(i)1}^{(1)} \overset{def}{=} \mathrm{K}_{11}^1 p_i^1$$

and

$$\underset{2}{N}{}_{(i)j}^{(1)} = \frac{h^{11}}{4}\left[\frac{\partial \overset{*}{g}_{ij}}{\partial x^k}\frac{\partial \overset{*}{H}}{\partial p_k^1} - \frac{\partial \overset{*}{g}_{ij}}{\partial p_k^1}\frac{\partial \overset{*}{H}}{\partial x^k} + \overset{*}{g}_{ik}\frac{\partial^2 \overset{*}{H}}{\partial x^j \partial p_k^1} + \overset{*}{g}_{jk}\frac{\partial^2 \overset{*}{H}}{\partial x^i \partial p_k^1}\right]. \tag{2.4}$$

Now, by a direct calculation, we obtain

$$\begin{aligned}
&\frac{\partial \overset{*}{g}_{ij}}{\partial x^k}\frac{\partial \overset{*}{H}}{\partial p_k^1} - \frac{\partial \overset{*}{g}_{ij}}{\partial p_k^1}\frac{\partial \overset{*}{H}}{\partial x^k} = 2h_{11}\left(1-2\delta_{ij}\right)p_i p_j\left(\sum_{k=1}^4 \frac{\sigma_k}{p_k}\right) + \\
&+4h_{11}\left(1-2\delta_{ij}\right)\left[-\frac{p_i p_j}{2}\left(\sum_{k=1}^4 \frac{\sigma_k}{p_k}\right) + \sigma_i p_j + \sigma_j p_i\right] = \\
&= 4h_{11}\left(1-2\delta_{ij}\right)\left(\sigma_i p_j + \sigma_j p_i\right)
\end{aligned}$$

and

$$\begin{aligned}
&\overset{*}{g}_{ik}\frac{\partial^2 \overset{*}{H}}{\partial x^j \partial p_k^1} + \overset{*}{g}_{jk}\frac{\partial^2 \overset{*}{H}}{\partial x^i \partial p_k^1} = \\
&= \sum_{k=1}^4\left\{\frac{e^{2\sigma(x)}\left(1-2\delta_{ik}\right)\left[\mathcal{P}^{1111}\right]^{-1/2}}{2}p_i^1 p_k^1 \cdot \frac{(-4)\,h_{11}e^{-2\sigma(x)}\sigma_j\left[\mathcal{P}^{1111}\right]^{1/2}}{p_k^1}\right\} + \\
&+ \sum_{k=1}^4\left\{\frac{e^{2\sigma(x)}\left(1-2\delta_{jk}\right)\left[\mathcal{P}^{1111}\right]^{-1/2}}{2}p_j^1 p_k^1 \cdot \frac{(-4)\,h_{11}e^{-2\sigma(x)}\sigma_i\left[\mathcal{P}^{1111}\right]^{1/2}}{p_k^1}\right\} = \\
&= \sum_{k=1}^4\left[-2h_{11}\left(1-2\delta_{ik}\right)\sigma_j p_i^1 - 2h_{11}\left(1-2\delta_{jk}\right)\sigma_i p_j^1\right] = -4h_{11}\left(\sigma_i p_j + \sigma_j p_i\right).
\end{aligned}$$

Consequently, we obtain

$$\underset{2}{N}{}_{(i)j}^{(1)} = -2\left(\sigma_i p_j + \sigma_j p_i\right)\delta_{ij} = -4\sigma_i p_i^1 \delta_{ij}.$$

Remark 2.2 Formula (2.4) of the spatial Hamiltonian canonical nonlinear connection is provided, via the Legendre duality, by the classical Lagrangian canonical nonlinear connection. For more details, the reader is invited to consult Miron's monograph ([7], pp. 166).

3 The Cartan canonical N-linear connection. Its d-torsions and d-curvatures

The nonlinear connection (2.3) produces the dual *adapted bases* of d-vector fields (no sum by i)

$$\left\{\frac{\delta}{\delta t}=\frac{\partial}{\partial t}-\mathrm{K}_{11}^{1}p_{r}^{1}\frac{\partial}{\partial p_{r}^{1}}\ ;\ \frac{\delta}{\delta x^{i}}=\frac{\partial}{\partial x^{i}}+4\sigma_{i}p_{i}^{1}\frac{\partial}{\partial p_{i}^{1}}\ ;\ \frac{\partial}{\partial p_{i}^{1}}\right\}\subset\mathcal{X}(E^{*}) \tag{3.1}$$

and d-covector fields (no sum by i)

$$\left\{dt\ ;\ dx^{i}\ ;\ \delta p_{i}^{1}=dp_{i}^{1}+\mathrm{K}_{11}^{1}p_{i}^{1}dt-4\sigma_{i}p_{i}^{1}dx^{i}\right\}\subset\mathcal{X}^{*}(E^{*}), \tag{3.2}$$

where $E^{*}=J^{1*}(\mathbb{R},M^{4})$. The naturalness of the geometrical adapted bases (3.1) and (3.2) is coming from the fact that, via a transformation of coordinates (1.3), their elements transform as the classical tensors. Therefore, the description of all subsequent geometrical objects on the dual 1-jet space $J^{1*}(\mathbb{R},M^{4})$ (e.g., the Cartan canonical linear connection, its torsion and curvature etc.) will be done in local adapted components. Consequently, by direct computations, we obtain the following geometrical result:

Proposition 3.1 The Cartan canonical N-linear connection produced by the (t,x)-conformal deformed Berwald-Moór Hamiltonian metric of order four (1.4) has the following adapted local components (no sum by i, j or k):

$$C\Gamma(N)=\left(\mathrm{K}_{11}^{1},\ A_{j1}^{i}=0,\ H_{jk}^{i}=4\delta_{j}^{i}\delta_{k}^{i}\sigma_{i},\ C_{i(1)}^{j(k)}=\mathsf{C}_{i}^{jk}\cdot\frac{p_{i}^{1}}{p_{j}^{1}p_{k}^{1}}\right), \tag{3.3}$$

where

$$\begin{aligned}\mathsf{C}_{i}^{jk} &= \frac{1-2\delta^{jk}-2\delta_{i}^{j}-2\delta_{i}^{k}+8\delta_{i}^{j}\delta_{i}^{k}}{8}=\\ &= \begin{cases}\dfrac{1}{8}, & i\neq j\neq k\neq i\\ -\dfrac{1}{8}, & i=j\neq k \text{ or } i=k\neq j \text{ or } j=k\neq i\\ \dfrac{3}{8}, & i=j=k.\end{cases}\end{aligned}$$

Proof. The adapted components of the Cartan canonical connection are given by the formulas (see [10]),

$$A_{j1}^{i}\overset{def}{=}\frac{\overset{*}{g}^{il}}{2}\frac{\delta\overset{*}{g}_{lj}}{\delta t}=0,\quad H_{jk}^{i}\overset{def}{=}\frac{\overset{*}{g}^{ir}}{2}\left(\frac{\delta\overset{*}{g}_{jr}}{\delta x^{k}}+\frac{\delta\overset{*}{g}_{kr}}{\delta x^{j}}-\frac{\delta\overset{*}{g}_{jk}}{\delta x^{r}}\right)=4\delta_{j}^{i}\delta_{k}^{i}\sigma_{i},$$

$$C_{i(1)}^{j(k)}\overset{def}{=}-\frac{\overset{*}{g}_{ir}}{2}\left(\frac{\partial\overset{*}{g}^{jr}}{\partial p_{k}^{1}}+\frac{\partial\overset{*}{g}^{kr}}{\partial p_{j}^{1}}-\frac{\partial\overset{*}{g}^{jk}}{\partial p_{r}^{1}}\right)=-\frac{\overset{*}{g}_{ir}}{2}\frac{\partial\overset{*}{g}^{jr}}{\partial p_{k}^{1}}.$$

Using the derivative operators (3.1), the direct calculations lead us to the required results.

Remark 3.2 It is important to note that the vertical d-tensor $C_{i(1)}^{j(k)}$ also has the properties (sum by m):

$$C_{i(1)}^{j(k)}=C_{i(1)}^{k(j)},\quad C_{i(1)}^{j(m)}p_{m}^{1}=0,\quad C_{m(1)}^{j(m)}=0,\quad C_{i(1)|m}^{j(m)}=0, \tag{3.4}$$

where

$$C^{l(k)}_{i(1)|j} \stackrel{def}{=} \frac{\delta C^{l(k)}_{i(1)}}{\delta x^j} + C^{r(k)}_{i(1)} H^l_{rj} - C^{l(k)}_{r(1)} H^r_{ij} + C^{l(r)}_{i(1)} H^k_{rj}.$$

Proposition 3.3 The Cartan canonical connection of the (t, x)-conformal deformed Berwald-Moór Hamiltonian metric of order four (1.4) has **two** effective local torsion d-tensors:

$$R^{(1)}_{(r)ij} = -4\sigma_{ij}\left(p^1_i \delta_{ir} - p^1_j \delta_{jr}\right), \quad P^{r(j)}_{i(1)} = \mathsf{C}^{rj}_i \cdot \frac{p^1_i}{p^1_r p^1_j},$$

where $\sigma_{ij} := \dfrac{\partial^2 \sigma}{\partial x^i \partial x^j}$.

Proof. A Cartan canonical connection on the dual 1-jet space $J^{1*}(\mathbb{R}, M^4)$ generally has *six* effective local d-tensors of torsion (for more details, see [10]). For the particular Cartan canonical connection (3.3) these reduce only to *two* (the other four are zero):

$$R^{(1)}_{(r)ij} \stackrel{def}{=} \frac{\delta \underset{2}{N}{}^{(1)}_{(r)i}}{\delta x^j} - \frac{\delta \underset{2}{N}{}^{(1)}_{(r)j}}{\delta x^i}, \quad P^{r(j)}_{i(1)} \stackrel{def}{=} C^{r(j)}_{i(1)}.$$

Proposition 3.4 The Cartan canonical connection of the (t, x)-conformal deformed Berwald-Moór Hamiltonian metric of order four (1.4) has **three** effective local curvature d-tensors:

$$R^l_{ijk} = \frac{\partial H^l_{ij}}{\partial x^k} - \frac{\partial H^l_{ik}}{\partial x^j} + H^r_{ij} H^l_{rk} - H^r_{ik} H^l_{rj} + C^{l(r)}_{i(1)} R^{(1)}_{(r)jk}, \quad P^{l\ (k)}_{ij(1)} = -C^{l(k)}_{i(1)|j},$$

$$S^{l(j)(k)}_{i(1)(1)} \stackrel{def}{=} \frac{\partial C^{l(j)}_{i(1)}}{\partial p^1_k} - \frac{\partial C^{l(k)}_{i(1)}}{\partial p^1_j} + C^{r(j)}_{i(1)} C^{l(k)}_{r(1)} - C^{r(k)}_{i(1)} C^{l(j)}_{r(1)}.$$

Proof. A Cartan canonical connection on the dual 1-jet space $J^{1*}(\mathbb{R}, M^4)$ generally has *five* effective local d-tensors of curvature (for all details, see [10]). For the particular Cartan canonical connection (3.3) these reduce only to *three* (the other two are zero). These are $S^{l(j)(k)}_{i(1)(1)}$ and

$$R^l_{ijk} \stackrel{def}{=} \frac{\delta H^l_{ij}}{\delta x^k} - \frac{\delta H^l_{ik}}{\delta x^j} + H^r_{ij} H^l_{rk} - H^r_{ik} H^l_{rj} + C^{l(r)}_{i(1)} R^{(1)}_{(r)jk},$$

$$P^{l\ (k)}_{ij(1)} \stackrel{def}{=} \frac{\partial H^l_{ij}}{\partial p^1_k} - C^{l(k)}_{i(1)|j} + C^{l(r)}_{i(1)} P^{(1)\ (k)}_{(r)j(1)} = -C^{l(k)}_{i(1)|j},$$

where

$$P^{(1)\ (k)}_{(r)j(1)} = \frac{\partial \underset{2}{N}{}^{(1)}_{(r)j}}{\partial p^1_k} + H^k_{rj} = 0.$$

4 From (t, x)-conformal deformations of the quartic Berwald-Moór Hamiltonian metric to field-like geometrical models

4.1 Momentum gravitational-like geometrical model

The (t, x)-conformal deformed Berwald-Moór Hamiltonian metric of order four (1.4) produces on the momentum phase space $J^{1*}(\mathbb{R}, M^4)$ the adapted metrical d-tensor (sum by i and j)

$$\mathbb{G} = h_{11} dt \otimes dt + \overset{*}{g}_{ij} dx^i \otimes dx^j + h_{11} \overset{*}{g}{}^{ij} \delta p^1_i \otimes \delta p^1_j, \tag{4.1}$$

where $\overset{*}{g}_{jk}$ and $\overset{*}{g}^{ij}$ are given by (2.2) and (2.1), and we have (no sum by i)

$$\delta p_i^1 = dp_i^1 + \kappa_{11}^1 p_i^1 dt - 4\sigma_i p_i^1 dx^i.$$

We believe that, from a physical point of view, the metrical d-tensor (4.1) may be regarded as a *"gravitational potential depending on momenta"*. In our abstract geometric-physical approach, one postulates that the momentum gravitational potential $\mathbb{G}$ is governed by the *geometrical Einstein equations*

$$\text{Ric } (C\Gamma(N)) - \frac{\text{Sc } (C\Gamma(N))}{2}\mathbb{G} = \mathcal{K}\mathbb{T}, \tag{4.2}$$

where

- Ric $(C\Gamma(N))$ is the *Ricci d-tensor* associated to the Cartan canonical linear connection (3.3); the Cartan canonical linear connection plays in our geometric-physical theory the same role as the Levi-Civita connection in the classical Riemannian theory of gravity;
- Sc $(C\Gamma(N))$ is the *scalar curvature*;
- $\mathcal{K}$ is the *Einstein constant* and $\mathbb{T}$ is an intrinsic *momentum stress-energy d-tensor of matter.*

Therefore, using the adapted basis of vector fields (3.1), we can locally describe the global geometrical Einstein equations (4.2). Consequently, some direct computations lead to:

Lemma 4.1 The Ricci tensor of the Cartan canonical connection of the (t, x)-conformal deformed Berwald-Moór Hamiltonian metric of order four (1.4) has the following **two** effective local Ricci d-tensors (no sum by i, j, k or l):

$$\begin{gathered} R_{ij} = \begin{cases} -2\sigma_{ij} - \dfrac{p_i^1}{p_k^1}\sigma_{jk} - \dfrac{p_i^1}{p_l^1}\sigma_{jl}, & i \neq j, \quad \{i,j,k,l\} = \{1,2,3,4\} \\ 0, & i = j, \end{cases} \\ R_{(1)(1)}^{(i)(j)} = -S_{(1)(1)}^{(i)(j)} = \frac{4\delta^{ij} - 1}{8}\frac{1}{p_i^1 p_j^1}. \end{gathered} \tag{4.3}$$

Proof. Generally, the Ricci tensor of a Cartan canonical connection $C\Gamma(N)$ (produced by an arbitrary momentum Hamiltonian function) is determined by *six* effective local Ricci d-tensors (for more details, see [11]). For the particular Cartan canonical connection (3.3) these reduce only to *two* (the other four are zero), where (sum by r and m):

$$R_{ij} \overset{def}{=} R_{ijm}^m = \frac{\partial H_{ij}^m}{\partial x^m} - \frac{\partial H_{im}^m}{\partial x^j} + H_{ij}^r H_{rm}^m - H_{im}^r H_{rj}^m + C_{i(1)}^{m(r)} R_{(r)jm}^{(1)},$$

$$\begin{aligned} S_{(1)(1)}^{(i)(j)} \overset{def}{=} S_{m(1)(1)}^{i(j)(m)} &= \frac{\partial C_{m(1)}^{i(j)}}{\partial p_m^1} - \frac{\partial C_{m(1)}^{i(m)}}{\partial p_j^1} + C_{m(1)}^{r(j)} C_{r(1)}^{i(m)} - C_{m(1)}^{r(m)} C_{r(1)}^{i(j)} = \\ &= \frac{\partial C_{m(1)}^{i(j)}}{\partial p_m^1} + C_{m(1)}^{r(j)} C_{r(1)}^{i(m)}. \end{aligned}$$

Lemma 4.2 The scalar curvature of the Cartan canonical connection of the (t, x)-conformal deformed Berwald-Moór Hamiltonian metric of order four (1.4) has the value

$$Sc\ (C\Gamma(N)) = -4e^{-2\sigma}\left[\mathcal{P}^{1111}\right]^{1/2}\Sigma_{11} - \frac{3}{2}h^{11}e^{2\sigma}\left[\mathcal{P}^{1111}\right]^{-1/2},$$

where
$$\Sigma_{11} = \frac{\sigma_{12}}{p_1^1 p_2^1} + \frac{\sigma_{13}}{p_1^1 p_3^1} + \frac{\sigma_{14}}{p_1^1 p_4^1} + \frac{\sigma_{23}}{p_2^1 p_3^1} + \frac{\sigma_{24}}{p_2^1 p_4^1} + \frac{\sigma_{34}}{p_3^1 p_4^1}.$$

Proof. The scalar curvature of the Cartan canonical connection (3.3) is given by the general formula Sc $(C\Gamma(N)) = \overset{*}{g}{}^{ij} R_{ij} - h^{11} \overset{*}{g}_{ij} S^{(i)(j)}_{(1)(1)}$.

The local description in the adapted basis of vector fields (3.1) of the global geometrical Einstein equations (4.2) leads us to

Proposition 4.3 The **geometrical Einstein-like equations** produced by the (t, x)-conformal deformed Berwald-Moór Hamiltonian metric of order four (1.4) are locally described by

$$\begin{cases} 2e^{-2\sigma} h_{11} \left[\mathcal{P}^{1111}\right]^{1/2} \Sigma_{11} + \dfrac{3}{4} e^{2\sigma} \left[\mathcal{P}^{1111}\right]^{-1/2} = \mathcal{K}\mathbb{T}_{11}, \\ R_{ij} + \left\{ 2e^{-2\sigma} \left[\mathcal{P}^{1111}\right]^{1/2} \Sigma_{11} + \dfrac{3}{4} h^{11} e^{2\sigma} \left[\mathcal{P}^{1111}\right]^{-1/2} \right\} \overset{*}{g}_{ij} = \mathcal{K}\mathbb{T}_{ij}, \\ -S^{(i)(j)}_{(1)(1)} + \left\{ 2e^{-2\sigma} h_{11} \left[\mathcal{P}^{1111}\right]^{1/2} \Sigma_{11} + \dfrac{3}{4} e^{2\sigma} \left[\mathcal{P}^{1111}\right]^{-1/2} \right\} \overset{*}{g}{}^{ij} = \mathcal{K}\mathbb{T}^{(i)(j)}_{(1)(1)}, \\ 0 = \mathbb{T}_{1i}, \quad 0 = \mathbb{T}_{i1}, \quad 0 = \mathbb{T}^{(i)}_{(1)1}, \\ 0 = \mathbb{T}_{1(1)}^{\ (i)}, \quad 0 = \mathbb{T}_{i(1)}^{\ (j)}, \quad 0 = \mathbb{T}^{(i)}_{(1)j}. \end{cases} \tag{4.4}$$

Corollary 4.4 The momentum stress-energy d-tensor of matter $\mathbb{T}$ satisfies the following **geometrical conservation-like laws** (sum by m):

$$\begin{cases} \mathbb{T}^1_{1/1} + \mathbb{T}^m_{1|m} + \mathbb{T}^{(1)}_{(m)1}\big|^{(m)}_{(1)} = 0 \\ \mathbb{T}^1_{i/1} + \mathbb{T}^m_{i|m} + \mathbb{T}^{(1)}_{(m)i}\big|^{(m)}_{(1)} = \mathsf{E}^m_{i|m} \\ \mathbb{T}^{1(i)}_{\ \ (1)/1} + \mathbb{T}^{m(i)}_{\ \ (1)|m} + \mathbb{T}^{(1)(i)}_{(m)(1)}\big|^{(m)}_{(1)} = \dfrac{e^{-2\sigma} \left[\mathcal{P}^{1111}\right]^{1/2}}{\mathcal{K}} \cdot \left[\dfrac{\Sigma_{11}}{p_i^1} + 2\dfrac{\partial \Sigma_{11}}{\partial p_i^1} \right], \end{cases}$$

where (sum by r):

$$\mathbb{T}^1_1 \overset{def}{=} h^{11}\mathbb{T}_{11} = \mathcal{K}^{-1} \left\{ 2e^{-2\sigma} \left[\mathcal{P}^{1111}\right]^{1/2} \Sigma_{11} + \frac{3}{4} h^{11} e^{2\sigma} \left[\mathcal{P}^{1111}\right]^{-1/2} \right\},$$

$$\mathbb{T}^m_1 \overset{def}{=} \overset{*}{g}{}^{mr}\mathbb{T}_{r1} = 0, \quad \mathbb{T}^{(1)}_{(m)1} \overset{def}{=} h^{11} \overset{*}{g}_{mr} \mathbb{T}^{(r)}_{(1)1} = 0, \quad \mathbb{T}^1_i \overset{def}{=} h^{11}\mathbb{T}_{1i} = 0,$$

$$\mathbb{T}^m_i \overset{def}{=} \overset{*}{g}{}^{mr}\mathbb{T}_{ri} = \mathsf{E}^m_i := \mathcal{K}^{-1} \cdot \left[\overset{*}{g}{}^{mr} R_{ri} + \delta^m_i \left\{ 2e^{-2\sigma} \left[\mathcal{P}^{1111}\right]^{1/2} \Sigma_{11} + \frac{3}{4} h^{11} e^{2\sigma} \left[\mathcal{P}^{1111}\right]^{-1/2} \right\} \right],$$

$$\mathbb{T}^{(1)}_{(m)i} \overset{def}{=} h^{11} \overset{*}{g}_{mr} \mathbb{T}^{(r)}_{(1)i} = 0, \quad \mathbb{T}^{1(i)}_{\ \ (1)} \overset{def}{=} h^{11}\mathbb{T}_{1(1)}^{\ (i)} = 0, \quad \mathbb{T}^{m(i)}_{\ \ (1)} \overset{def}{=} \overset{*}{g}{}^{mr}\mathbb{T}_{r(1)}^{\ (i)} = 0,$$

$$\mathbb{T}^{(1)(i)}_{(m)(1)} \overset{def}{=} h^{11} \overset{*}{g}_{\ mr} \mathbb{T}^{(r)(i)}_{(1)(1)} = \frac{h^{11} e^{2\sigma} \left[\mathcal{P}^{1111}\right]^{-1/2}}{8\mathcal{K}} \frac{p_m}{p_i} +$$
$$+\delta^i_m \cdot \left[\frac{h^{11} e^{2\sigma} \left[\mathcal{P}^{1111}\right]^{-1/2}}{4\mathcal{K}} + \frac{2e^{-2\sigma} \left[\mathcal{P}^{1111}\right]^{1/2} \Sigma_{11}}{\mathcal{K}} \right],$$

and we also have (summation by m and r, but no sum by i)

$$\mathbb{T}^1_{1/1} \stackrel{def}{=} \frac{\delta \mathbb{T}^1_1}{\delta t} + \mathbb{T}^1_1 \mathrm{K}^1_{11} - \mathbb{T}^1_1 \mathrm{K}^1_{11} = \frac{\delta \mathbb{T}^1_1}{\delta t}, \quad \mathbb{T}^m_{1|m} \stackrel{def}{=} \frac{\delta \mathbb{T}^m_1}{\delta x^m} + \mathbb{T}^r_1 H^m_{rm},$$

$$\mathbb{T}^{(1)}_{(m)1}|^{(m)}_{(1)} \stackrel{def}{=} \frac{\partial \mathbb{T}^{(1)}_{(m)1}}{\partial p^1_m} - \mathbb{T}^{(1)}_{(r)1} C^{r(m)}_{m(1)} = \frac{\partial \mathbb{T}^{(1)}_{(m)1}}{\partial p^1_m},$$

$$\mathbb{T}^1_{i/1} \stackrel{def}{=} \frac{\delta \mathbb{T}^1_i}{\delta t} + \mathbb{T}^1_i \mathrm{K}^1_{11} - \mathbb{T}^1_r A^r_{i1} = \frac{\delta \mathbb{T}^1_i}{\delta t} + \mathbb{T}^1_i \mathrm{K}^1_{11},$$

$$\mathbb{T}^m_{i|m} \stackrel{def}{=} \frac{\delta \mathbb{T}^m_i}{\delta x^m} + \mathbb{T}^r_i H^m_{rm} - \mathbb{T}^m_r H^r_{im} = \mathsf{E}^m_{i|m} := \frac{\delta \mathsf{E}^m_i}{\delta x^m} + 4\mathsf{E}^m_i \sigma_m - 4\mathsf{E}^i_i \sigma_i,$$

$$\mathbb{T}^{(1)}_{(m)i}|^{(m)}_{(1)} \stackrel{def}{=} \frac{\partial \mathbb{T}^{(1)}_{(m)i}}{\partial p^1_m} - \mathbb{T}^{(1)}_{(r)i} C^{r(m)}_{m(1)} - \mathbb{T}^{(1)}_{(m)r} C^{r(m)}_{i(1)},$$

$$\mathbb{T}^{1(i)}_{(1)/1} \stackrel{def}{=} \frac{\delta \mathbb{T}^{1(i)}_{(1)}}{\delta t} + \mathbb{T}^{1(r)}_{(1)} A^i_{r1}, \quad \mathbb{T}^{m(i)}_{(1)|m} \stackrel{def}{=} \frac{\delta \mathbb{T}^{m(i)}_{(1)}}{\delta x^m} + 4\mathbb{T}^{m(i)}_{(1)} \sigma_m + 4\mathbb{T}^{i(i)}_{(1)} \sigma_i,$$

$$\mathbb{T}^{(1)(i)}_{(m)(1)}|^{(m)}_{(1)} \stackrel{def}{=} \frac{\partial \mathbb{T}^{(1)(i)}_{(m)(1)}}{\partial p^1_m} - \mathbb{T}^{(1)(i)}_{(r)(1)} C^{r(m)}_{m(1)} + \mathbb{T}^{(1)(r)}_{(m)(1)} C^{i(m)}_{r(1)}.$$

Proof. The local Einstein equations (4.4), together with some direct computations, lead us to what we were looking for.

4.2 Momentum electromagnetic-like geometrical model

In the paper [11], a geometrical theory for an electromagnetism depending on momenta was also created, using only a given Hamiltonian function H on the momentum phase space $J^{1*}(\mathbb{R}, M^4)$. In the background of the jet momentum Hamiltonian geometry from this paper, we work with the *electromagnetic distinguished* 2*-form* $\mathbb{F} = F^{(i)}_{(1)j} \delta p^1_i \wedge dx^j$, where (sum by r and m)

$$F^{(i)}_{(1)j} = \frac{h^{11}}{2} \left[\overset{*}{g}{}^{jr} \underset{2}{N}{}^{(1)}_{(r)i} - \overset{*}{g}{}^{ir} \underset{2}{N}{}^{(1)}_{(r)j} + \left(\overset{*}{g}{}^{jr} H^m_{ri} - \overset{*}{g}{}^{ir} H^m_{rj} \right) p^1_m \right].$$

The above electromagnetic components depending on momenta are characterized by some natural *geometrical Maxwell-like equations* (for more details, see Oană and Neagu [10], [11]).

By a direct calculation, we see that the (t, x)-conformal deformed Berwald-Moór Hamiltonian metric of order four (1.4) produces null momentum electromagnetic components: $F^{(i)}_{(1)j} = 0$. Consequently, our dual jet (t, x)-conformal deformed Berwald-Moór Hamiltonian geometrical electromagnetic theory is trivial one. Probably, this fact suggests that the dual jet (t, x)-conformal deformed Berwald-Moór Hamiltonian structure (1.4) has rather gravitational connotations than electromagnetic ones on the momentum phase space $J^{1*}(\mathbb{R}, M^4)$.

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СТРУЙНАЯ ГЕОМЕТРИЗАЦИЯ РИМАНА-ГАМИЛЬТОНА ДЛЯ ЗАВИСЯЩЕЙ ОТ ИМПУЛЬСА КОНФОРМНО-ДЕФОРМИРОВАННОЙ МЕТРИКИ БЕРВАЛЬДА-МООРА ЧЕТВЕРТОЙ СТЕПЕНИ

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В настоящей статье мы предлагаем вниманию читателя (d-)риманову геометрию (в смысле d-связности, d-вращения, d-кривизны и некоторых геометрических моделей гравитационного и электромагнитного типа) в дуальном 1-струйном пространстве $J^{1*}(\mathbb{R}, M^4)$ для (t, x)-конформной деформированной метрики Бервальда-Моора четвертой степени.

Ключевые слова: (t, x)-конформная деформация метрики Бервальда-Моора четвертой степени, каноническая нелинейная связность, каноническая линейная связность Картана, d-вращение и d-кривизна, геометрические уравнения Эйнштейновского типа.