ON THE FINSLERIAN MECHANICAL SYSTEMS

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The notion of Finslerian Mechanical Systems was been introduced by author as a triple $\Sigma_F = (M, \mathcal{E}_F, Fe)$ formed by configuration space M, kinetic energy $\mathcal{E}_F$ of a semidefinite Finsler space $F^n = (M, F)$ and the external forces Fe. Fundamental equations of Σ_F are the Lagrange equations. One determines the canonical semispray S and proves that the integral curves of S are the evolution curves of Σ_F. Thus, the geometrical theory of the Finslerian mechanical systems Σ_F can be studied by means of dynamical systems S on the velocity space TM.

Key Words: semidefinite Finsler space, Finslerian mechanical systems.

Introduction

My lecture to The VIIIth International Conference "Finsler Extensions of Relativity Theory", Moscow, July–August, 2012, is a survey on the Analytical Mechanics of Finslerian Mechanics, introduced by author in the papers [5, 12, 16, 17, 21]. These systems are defined by a triple $\Sigma_F = (M, F^2, Fe)$ where M is the configuration space, $F(x, y)$ is the fundamental function of a semidefinite Finsler space $F^n = (M, F(x, y))$ and $Fe(x, y)$ are the external forces. Of course, F^2 is the kinetic energy of the space. The fundamental equations are the Lagrange equations:

$$E_i(F^2) \equiv \frac{d}{dt}\frac{\partial F^2}{\partial \dot{x}^i} - \frac{\partial F^2}{\partial x^i} = F_i(x, \dot{x}).$$

We study here the canonical semispray S of Σ_F and the geometry of the pair (TM, S), where TM is velocity space, [17].

One obtain a generalization of the theory of Riemannian Mechanical Systems in the non-conservative case. It has numerous applications and justifies the introduction of such new kind of Analytical Mechanics.

1 Semidefinite Finsler spaces

Definition 1.1 A Finsler space with semidefinite Finsler metric is a pair $F^n = (M, F(x, y))$ where the function $F : TM \to \mathbb{R}$ satisfies the following axioms:

1° F is differentiable on $\widetilde{TM}$ and continuous on the null section of $\pi : TM \to M$;

2° $F \geq 0$ on TM;

3° F is positive 1-homogeneous with respect to velocities $\dot{x}^i = y^i$.

4° The fundamental tensor $g_{ij}(x, y)$

$$g_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j} \tag{1.1}$$

has a constant signature on $\widetilde{TM}$;

5° The Hessian of fundamental function F^2 with elements $g_{ij}(x, y)$ is nonsingular:

$$\det(g_{ij}(x, y)) \neq 0 \text{ on } \widetilde{TM}. \tag{1.2}$$

Example. If $g_{ij}(x)$ is a semidefinite Riemannian metric on M, then

$$F = \sqrt{|g_{ij}(x)y^i y^j|} \tag{1.3}$$

is a function with the property $F^n = (M, F)$ is a semidefinite Finsler space.

Any Finsler space $F^n = (M, F(x, y))$, in the sense of definition 1.1, is a definite Finsler space. In this case the property 5° is automatical verified.

But, these two kind of Finsler spaces have a lot of common properties. Therefore, we will speak in general on Finsler spaces. The following properties hold:

1° The fundamental tensor $g_{ij}(x, y)$ is 0-homogeneous;

2° $F^2 = g_{ij}(x, y)y^i y^j$;

3° $p_i = \dfrac{1}{2}\dfrac{\partial F^2}{\partial y^i}$ is $d-$covariant vector field;

4° The Cartan tensor

$$C_{ijk} = \frac{1}{4}\frac{\partial^3 F^2}{\partial y^i \partial y^j \partial y^k} = \frac{1}{2}\frac{\partial g_{ij}}{\partial y^k} \tag{1.4}$$

is totally symmetric and

$$y^i C_{ijk} = C_{0jk} = 0. \tag{1.5}$$

5° $\omega = p_i dx^i$ is 1-form on $\widetilde{TM}$ (the Cartan 1-form);

6° $\theta = d\omega = dp_i \wedge dx^i$ is 2-form (the Cartan 2-form);

7° The Euler-Lagrange equations of F^n are

$$E_i(F^2) = \frac{d}{dt}\frac{\partial F^2}{\partial y^i} - \frac{\partial F^2}{\partial x^i} = 0, \quad y^i = \frac{dx^i}{dt} \tag{1.6}$$

8° The energy $\mathcal{E}_F$ of F^n is

$$\mathcal{E}_F = y^i \frac{\partial F^2}{\partial y^i} - F^2 = F^2 \tag{1.7}$$

9° The energy $\mathcal{E}_F$ is conserved along to every integral curve of Euler-Lagrange equations (1.6);

10° In the canonical parametrization, the equations (1.6) give the geodesics of F^n;

11° The Euler-Lagrange equations (1.6) can be written in the equivalent form

$$\frac{d^2x^i}{dt^2} + \gamma^i_{jk}\left(x, \frac{dx}{dt}\right)\frac{dx^j}{dt}\frac{dx^k}{dt} = 0, \tag{1.8}$$

where $\gamma^i_{jk}\left(x, \dfrac{dx}{dt}\right)$ are the Christoffel symbols of the fundamental tensor $g_{ij}(x, y)$.

12° The canonical semispray S is

$$S = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y)\frac{\partial}{\partial y^i} \tag{1.9}$$

with the coefficients:

$$2G^i(x, y) = \gamma^i_{jk}(x, y)y^j y^k = \gamma^i_{00}(x, y), \tag{1.9'}$$

(the index "0" means the contraction with y^i).

13° The canonical semispray S is 2-homogeneous with respect to y^i. So, S is a spray.

14° The nonlinear connection N determined by S is also canonical and it is exactly the famous Cartan nonlinear connection of the space F^n. Its coefficients are

$$N^i{}_j(x,y) = \frac{\partial G^i(x,y)}{\partial y^j} = \frac{1}{2}\frac{\partial}{\partial y^j}\gamma^i_{00}(x,y). \tag{1.10}$$

An equivalent form for the coefficients N^i_j is as follows

$$N^i{}_j = \gamma^i_{j0}(x,y) - C^i_{jk}(x,y)\gamma^k_{00}(x,y). \tag{1.10'}$$

Consequently, we have

$$N^i{}_0 = \gamma^i_{00} = 2G^i. \tag{1.11}$$

Therefore, we can say: The semispray S' determined by the Cartan nonlinear connection N is the canonical spray S of space F^n.

15° The Cartan nonlinear connection N determines a splitting of vector space T_uTM, $\forall\ u \in\ TM$ of the form:

$$T_uTM = N_u \oplus V_u, \quad \forall u \in TM \tag{1.12}$$

Thus, the adapted basis $\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\right)$, $(i = 1,..,n)$, to the previous splitting has the local vector fields $\frac{\delta}{\delta x^i}$ given by:

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^j{}_i(x,y)\frac{\partial}{\partial y^j}, \quad i = 1,..,n, \tag{1.13}$$

with the coefficients $N^i{}_j(x,y)$ from (1.6).
Its dual basis is $(dx^i, \delta y^i)$, where

$$\delta y^i = dy^i + N^i{}_j(x,y)dx^j. \tag{1.14}$$

The autoparallel curves of the nonlinear connection N are given by, [5, 18],

$$\frac{d^2x^i}{dt^2} + N^i{}_j\left(x, \frac{dx}{dt}\right)\frac{dx^j}{dt} = 0. \tag{1.15}$$

Using the dynamic derivative ∇ defined by N, the equations (1.11) can be written as follows

$$\nabla\left(\frac{dx^i}{dt}\right) = 0. \tag{1.11'}$$

16° The variational equations of the autoparallel curves (1.11) give the Jacobi equations:

$$\nabla^2\xi^i + \left(\frac{\partial N^i{}_j}{\partial y^k}\frac{dx^j}{dt} - N^i{}_k\right)\nabla\xi^k + R^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt} = 0. \tag{1.16}$$

The vector field $\xi^i(t)$ along a solution $c(t)$ of the equations (1.11) and which verifies the previous equations is called a Jacobi field. In the Riemannian case, $\frac{\partial g_{ij}}{\partial y^k} = 0$, the Jacobi equations (1.12) are exactly the classical Jacobi equations:

$$\nabla^2\xi^i + R^i_{jlk}(x)\frac{dx^l}{dt}\frac{dx^j}{dt}\xi^k = 0 \tag{1.17}$$

17° A distinguished metric connections D with the coefficients $C\Gamma(N) = \left(F^i_{jk}, C^i_{jk}\right)$ is defined as a N-linear connection on TM, metric with respect to the fundamental tensor $g_{ij}(x,y)$ of Finsler space F^n, i.e. we have

$$\begin{aligned} g_{ij|k} &= \frac{\delta g_{ij}}{\delta x^k} - F^s_{ik} g_{sj} - F^s_{jk} g_{is} = 0, \\ g_{ij}|_k &= \frac{\partial g_{ij}}{\partial y^k} - C^s_{ik} g_{sj} - C^s_{jk} g_{is} = 0. \end{aligned} \tag{1.18}$$

18° The following theorem holds:

Theorem 1.1

1° *There is an unique N-linear connection D, with coefficients $C\Gamma(N)$ which satisfies the following system of axioms: A_1. N is the Cartan nonlinear connection of Finsler space F^n.*

A_2. D is metrical, (i.e. D satisfies (2.1.14)*).*

A_3. $T^i_{jk} = F^i_{jk} - F^i_{kj} = 0$, $S^i_{jk} = C^i_{jk} - C^i_{kj} = 0$.

2° *The metric N-linear connection D has the coefficients $C\Gamma(N) = \left(F^i_{jk}, C^i_{jk}\right)$ given by the generalized Christoffel symbols*

$$\begin{aligned} F^i_{jk} &= \frac{1}{2} g^{is} \left(\frac{\delta g_{sj}}{\delta x^k} + \frac{\delta g_{sk}}{\delta x^j} - \frac{\delta g_{jk}}{\delta x^s} \right), \\ C^i_{jk} &= \frac{1}{2} g^{is} \left(\frac{\partial g_{sj}}{\partial y^k} + \frac{\partial g_{sk}}{\partial y^j} - \frac{\partial g_{jk}}{\partial y^s} \right). \end{aligned} \tag{1.19}$$

20° By means of this theorem, it is not difficult to see that we have

$$C^i_{jk} = g^{is} C_{sjk} \tag{1.20}$$

and

$$y^i{}_{|k} = 0. \tag{1.21}$$

The Cartan nonlinear connection N determines on $\widetilde{TM}$ an almost complex structure $\mathbb{F}$, as follows:

$$\mathbb{F}\left(\frac{\delta}{\delta x^i}\right) = -\frac{\partial}{\partial y^i}, \quad \mathbb{F}\left(\frac{\partial}{\partial y^i}\right) = \frac{\delta}{\delta x^i}, \qquad i = 1,..,n. \tag{1.22}$$

But one can see that $\mathbb{F}$ is the tensor field on $\widetilde{TM}$:

$$\mathbb{F} = -\frac{\partial}{\partial y^i} \otimes dx^i + \frac{\delta}{\delta x^i} \otimes \delta y^i, \tag{1.22'}$$

with the 1-forms δy^i and the vector field $\dfrac{\delta}{\delta x^i}$ given by (1.10), (1.9), (1.6).

It is not difficult to prove that: The almost complex structure $\mathbb{F}$ is integrable if and only if the distribution N is integrable on TM.

22° The Sasaki-Matsumoto lift of the fundamental tensor g_{ij} of Finsler space F^n is

$$\mathbb{G}(x,y) = g_{ij}(x,y) dx^i \otimes dx^j + g_{ij}(x,y) \delta y^i \otimes \delta y^j. \tag{1.23}$$

The tensor field $\mathbb{G}$ determines a pseudo-Riemannian structure on TM.

23° The following theorem is known:

Theorem 1.2.

1° *The pair* $(\mathbb{G}, \mathbb{F})$ *is an almost Hermitian structure on* $\widetilde{TM}$ *determined only by the Finsler space* F^n.

2° *The symplectic structure associate to the structure* $(\mathbb{G}, \mathbb{F})$ *is the Cartan 2-form:*

$$\theta = 2g_{ij}\delta y^i \wedge dx^j. \tag{1}$$

3° *The space* $(\widetilde{TM}, \mathbb{G}, \mathbb{F})$ *is almost Kählerian.*

The space $H^{2n} = (\widetilde{TM}, \mathbb{G}, \mathbb{F})$ is called *the almost Kählerian model* of the Finsler space F^n.

G.S. Asanov in the paper [5] proved that the metric $\mathbb{G}$ from (1.23) does not satisfies the principle of the Post-Newtonian Calculus. This is due to the fact that the horizontal and vertical terms of $\mathbb{G}$ do not have the same physical dimensions.

This is the reason for R. Miron to introduce a new lift of the fundamental tensor g_{ij}, [5, 17, 18], in the form:

$$\widetilde{\mathbb{G}}(x, y) = g_{ij}(x, y)dx^i \otimes dx^j + \frac{a^2}{||y||^2}g_{ij}(x, y)\delta y^i \otimes \delta y^j$$

where $a > 0$ is a real constant imposed by applications in Theoretical Physics and where $\|y\|^2 = g_{ij}(x, y)y^i y^j = F^2$ has the property $F^2 > 0$. The lift $\mathbb{G}$ is 2-homogeneous with respect to y^i. The Sasaki-Matsumoto lift $\mathbb{G}$ has not the property of homogeneity, [8, 21].

Two examples:

1. *Randers spaces.* They have been defined by R. S. Ingarden as a triple $RF^n = (M, \alpha+\beta, N)$, where $\alpha + \beta$ is a Randers metric and N is the Cartan nonlinear connection of the Finsler space $F^n = (M, \alpha + \beta)$, [13].

2. *Ingarden spaces.* These spaces have been defined by R. Miron, [5, 18], as a triple $IF^n = (M, \alpha+\beta, N_L)$, where $\alpha+\beta$ is a Randers metric and N_L is the Lorentz nonlinear connection of $F^n = (M, \alpha + \beta)$ having the coefficients

$$N^i_j(x, y) = \overset{\circ}{\gamma}{}^i_{jk}(x)y^k - \overset{\circ}{F}{}^i_j(x), \qquad \overset{\circ}{F}{}^i_j = \frac{1}{2}a^{is}(x)\left(\frac{\partial b_s}{\partial x^j} - \frac{\partial b_j}{\partial x^s}\right). \tag{2}$$

The Christoffel symbols are constructed with the Riemannian metric tensor $a_{ij}(x)$ of the Riemann space (M, α^2) and $\overset{\circ}{F}{}^i_j(x)$ is the electromagnetic tensor determined by the electromagnetic form $(\alpha + \beta)$.

2 The notion of Finslerian mechanical system

As we know [5, 18], the Riemannian mechanical systems $\Sigma_{\mathcal{R}} = (M, T, Fe)$ is defined as a triple in which M is the configuration space, T is the kinetic energy and Fe are the external forces, which depend on the material point $x \in M$ and depend on velocities $y^i = \dfrac{dx^i}{dt}$.

Extending the previous ideas, we introduce the notion of Finslerian Mechanical System, studied by author in the paper [17]. The shortly theory of this analytical mechanics can be find in the joint book *Finsler-Lagrange Geometry. Applications to Dynamical Systems*, by Ioan Bucataru and Radu Miron, Romanian Academy Press, Bucharest, 2007.

In a different manner, M. de Leon and colab. [11], M. Crampin et colab. [17, 18], have studied such kind of new Mechanics. The time dependent case is considered in the book [4].

A Finslerian mechanical system Σ_F is defined as a triple

$$\Sigma_F = (M, \mathcal{E}_{F^2}, Fe) \tag{2.1}$$

where M is a real differentiable manifold of dimension n, called *the configuration space*, $\mathcal{E}_{F^2}$ is *the energy* of an a priori given Finsler space $F^n = (M, F(x,y))$, which can be positive defined or semidefined, and $Fe(x,y)$ are the external forces given as a vertical vector field on the tangent manifold TM. We continue to say that TM is the velocity space of M.

Evidently, the Finslerian mechanical system Σ_F is a straightforward generalization of the known notion of Riemannian mechanical system $\Sigma_{\mathcal{R}}$ obtained for $\mathcal{E}_{F^2}$ as kinetic energy of a Riemann space $\mathcal{R}^n = (M, g)$.

Therefore, we can introduce the evolution (or fundamental) equations of Σ_F by means of the following Postulate:

Postulate. *The evolution equations of the Finslerian mechanical system Σ_F are the Lagrange equations:*

$$\frac{d}{dt}\frac{\partial \mathcal{E}_{F^2}}{\partial y^i} - \frac{\partial \mathcal{E}_{F^2}}{\partial x^i} = F_i(x,y), \qquad y^i = \frac{dx^i}{dt} \tag{2.2}$$

where the energy is

$$\mathcal{E}_{F^2} = y^i \frac{\partial F^2}{\partial y^i} - F^2 = F^2, \tag{2.3}$$

and $F_i(x,y)$, $(i = 1,..,n)$, are the covariant components of the external forces Fe:

$$\begin{cases} Fe(x,y) = F^i(x,y)\dfrac{\partial}{\partial y^i} \\ \\ F_i(x,y) = g_{ij}(x,y)F^i(x,y), \end{cases} \tag{2.4}$$

and

$$g_{ij}(x,y) = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}, \quad \det(g_{ij}(x,y)) \neq 0, \tag{2.5}$$

is the fundamental (or metric) tensor of Finsler space $F^n = (M, F(x,y))$.

Finally, the Lagrange equations of the Finslerian mechanical system are:

$$\frac{d}{dt}\frac{\partial F^2}{\partial y^i} - \frac{\partial F^2}{\partial x^i} = F_i(x,y), \qquad y^i = \frac{dx^i}{dt}. \tag{2.6}$$

A more convenient form of the previous equations is given by:

Theorem 2.1. *The Lagrange equations* (2.6) *are equivalent to the second order differential equations:*

$$\frac{d^2x^i}{dt^2} + \gamma^i_{jk}\left(x, \frac{dx}{dt}\right)\frac{dx^j}{dt}\frac{dx^k}{dt} = \frac{1}{2}F^i\left(x, \frac{dx}{dt}\right), \tag{2.7}$$

Proof. Writing the kinetic energy $F^2(x,y)$ in the form:

$$F^2(x,y) = g_{ij}(x,y)y^i y^j, \tag{2.8}$$

the equivalence of the systems of equations (2.6) and (2.7) is not difficult to establish.

But, the form (2.7) is very convenient in applications. So, we obtain a first result expressed in the following theorems:

Theorem 2.2. *The trajectories of the Finslerian mechanical system Σ_F, without external forces ($Fe \equiv 0$), are the geodesics of the Finsler space F^n.*

Indeed, $F^i(x,y) \equiv 0$ and the SODE (2.7) imply the equations (2.4) of geodesics of space F^n.

A second important result is a consequence of the Lagrange equations, too.

Theorem 2.3. *The variation of kinetic energy $\mathcal{E}_{F^2} = F^2$ of the mechanical system Σ_F along the evolution curves (2.6) is given by*

$$\frac{d\mathcal{E}_{F^2}}{dt} = \frac{dx^i}{dt} F_i. \tag{2.9}$$

Theorem 2.4. *The kinetic energy $\mathcal{E}_{F^2}$ of the system Σ_F is conserved along the evolution curves* (2.6) *if the external forces Fe are orthogonal to the evolution curves.*

The external forces Fe are called *dissipative* if the scalar product $\langle \mathbb{C}, Fe \rangle$ is negative, [17, 18].

Theorem 2.5. *The kinetic energy $\mathcal{E}_{F^2}$ decreases along the evolution curves of the Finslerian mechanical system Σ_F if and only if the external forces Fe are dissipative.*

Some examples of Finslerian mechanical systems

1° The systems $\Sigma_F = (M, \mathcal{E}_{F^2}, Fe)$ given by $F^n = (M, \alpha + \beta)$ as a Randers space and $Fe = \beta\mathbb{C} = \beta y^i \dfrac{\partial}{\partial y^i}$. Evidently Fe is 2-homogeneous with respect to y^i.

2° Σ_F determined by $F^n = (M, \alpha + \beta)$ and $Fe = \alpha\mathbb{C}$.

3° Σ_F with $F^n = (M, \alpha + \beta)$ and $Fe = (\alpha + \beta)\mathbb{C}$.

4° Σ_F defined by a Finsler space $F^n = (M, F)$ and $Fe = a^i_{jk}(x) y^j y^k \dfrac{\partial}{\partial y^i}$, $a^i_{jk}(x)$ being a symmetric tensor on the configuration space M of type $(1, 2)$.

3 The evolution semispray of the system Σ_F

The Lagrange equations (2.6) give us the integral curves of a remarkable semispray on the velocity space TM, which governed the geometry of Finslerian mechanical system Σ_F. So, if the external forces Fe are global defined on the manifold TM, we obtain:

Theorem 3.1. [Miron, [17, 18]] *For the Finslerian mechanical systems Σ_F, the following properties hold good:*

1° *The operator S defined by*

$$S = y^i \frac{\partial}{\partial x^i} - \left(2 \overset{\circ}{G}{}^i - \frac{1}{2} F^i \right) \frac{\partial}{\partial y^i}; \qquad 2 \overset{\circ}{G}{}^i = \gamma^i_{jk}(x, y) y^j y^k \tag{3.1}$$

is a vector field, global defined on the phase space TM.

2° *S is a semispray which depends only on Σ_F and it is a spray if Fe is 2-homogeneous with respect to y^i.*

3° *The integral curves of the vector field S are the evolution curves given by the Lagrange equations* (2.7) *of Σ_F.*

Proof.

1° Let us consider the canonical semispray $\overset{\circ}{S}$ of the Finsler space F^n. Thus from (2.3.1) we have

$$S = \overset{\circ}{S} + \frac{1}{2} Fe. \tag{3.2}$$

It follows that S is a vector field on TM.

2° Since Fe is a vertical vector field, then S is a semispray. Evidently, S depends on Σ_F, only.

3° The integral curves of S are given by:

$$\frac{dx^i}{dt} = y^i; \quad \frac{dy^i}{dt} + 2 \overset{\circ}{G^i} (x, y) = \frac{1}{2} F^i(x, y). \tag{3.3}$$

The previous system of differential equations is equivalent to system (2.7).

In the book of I. Bucataru and R. Miron [5], one proves the following important result, which extend a known J. Klein theorem, [9]:

Theorem 3.2. *The semispray S, given by the formula* (3.1)*, is the unique vector field on $\widetilde{TM}$, solution of the equation:*

$$i_S \overset{\circ}{\omega} = -dT + \sigma, \tag{3.4}$$

where $\overset{\circ}{\omega}$ is the symplectic structure of the Finsler space $F^n = (M, F)$, $T = \frac{1}{2}F^2 = \frac{1}{2} g_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt}$ and σ is the 1-form of external forces:

$$\sigma = F_i(x, y) dx^i. \tag{3.5}$$

In the terminology of J. Klein, [9], S is the dynamical system of Σ_F, defined on the tangent manifold TM. We will say that S is the evolution semispray of Σ_F.

By means of semispray S we can develop the geometry of the Finslerian mechanical system Σ_F. So, all geometrical notion derived from S, as nonlinear connections, N-linear connections etc. will be considered as belong to the system Σ_F. But, all this construction is developed in the papers [17, 18]. A good application can be found in the Pavlov and Kokarev's paper [23].

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О ПОЛУОПРЕДЕЛЕННЫХ ФИНСЛЕРОВЫХ МЕХАНИЧЕСКИХ СИСТЕМАХ

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Понятие финслеровых механических систем было введено автором, как триплет $\Sigma_F = (M, \mathcal{E}_F, Fe)$ формируемый конфигурационным пространством M, кинетической энергией $\mathcal{E}_F$ полуопределенного финслерова пространства $F^n = (M, F)$ и внешней силой Fe. Фундаментальные уравнения Σ_F являются уравнениями Лагранжа. Можно определить канонический полуспрей S и доказать, что интегральные кривые S являются динамическими кривыми Σ_F. Таким образом, геометрическая теория финслеровых динамических систем Σ_F может изучаться при помощи динамических систем S в пространстве скоростей TM.

Ключевые слова: полуопределенное финслерово пространство, финслеровы механические системы.