EQUATIONS OF ELECTROMAGNETISM IN SOME SPECIAL ANISOTROPIC SPACES. PART 2

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By using variational calculus and exterior derivative formalism, we proposed in [1] a new geometric approach to electromagnetism in spaces with metrics obtained as small deformations of flat Finsler metrics. The ideas were extended to general Finsler spaces in [11].

In the present paper, we provide more details regarding generalized currents, the domain of integration and gauge invariance. Also, for flat Finsler spaces, we define the generalized energy-momentum tensor as the symmetrized Noether current corresponding to the invariance of the field Lagrangian with respect to spacetime translations.

Key Words: Finsler space, tangent bundle, electromagnetic tensor, electromagnetic potential, stress-energy-momentum tensor.
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1 Introduction

In Finsler spaces, the metric tensor depends on the directional variables. Since Maxwell equations involve the metric tensor, their solutions basically also depend on these. Hence, the electromagnetic tensor (and the corresponding 4-potential) depend both on positional and on directional variables, meaning that they are no longer defined on the spacetime manifold M, but on its tangent bundle TM. The dependence of the 4-potential on the directional variables leads to new terms in the equations of motion of charged particles and in the expression of currents. In the picture on the whole TM, together with the usual Maxwell equations there appear new equations and a vertical counterpart of the usual 4-current (which provides the horizontal part of a vector field J on TM). By applying the classical procedure based on Noether's theorem, we also obtain a generalization of the notion of stress-energy tensor of the electromagnetic field. The obtained tensor consists of two blocks, one of which is similar to the usual energy-momentum tensor.

Our approach is based on variational calculus and classical methods in theoretical physics (adapted to the tangent bundle). Also, we used the language of differential forms, which is naturally related to variational calculus and offers the possibility of a concise and elegant writing of equations.

2 A brief overview of the Riemannian case

Let us consider as spacetime manifold, a pseudo-Riemannian manifold (M, g) of dimension 4 and denote local coordinates on M by $x = (x^i)_{i=\overline{0,3}}$. The first coordinate is regarded as the time coordinate and $(x^\alpha)_{\alpha=\overline{1,3}}$, as spatial coordinates. The metric $g = g(x)$ is supposed to have Lorentz signature $(-,+,+,+)$. Here are some other notations and conventions we will use in the following:

- Latin indices $i, j, k, ...$ take values from 0 to 3; Greek indices $\alpha, \beta, \gamma, ...$ take values from 1 to 3;

- $_{,k}$ – partial derivative with respect to $\dfrac{\partial}{\partial x^k}$;

- $_{|k}$ – Levi-Civita covariant derivative with respect to $\dfrac{\partial}{\partial x^k}$; $\gamma^i{}_{jk} = \gamma^i{}_{jk}(x)$ - Christoffel symbols of g;

- $g = \det(g_{ij})$; usually, it will be clear from the context whether we refer to g as the metric tensor or to the determinant of the corresponding matrix;
- $\flat : TM \to T^*M$, $\sharp : T^*M \to TM$ – musical isomorphisms (lowering indices of vector fields/raising indices of 1-forms);
- d - exterior derivative of differential forms;
- $d^4x = dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$, $\quad d^3x = dx^1 \wedge dx^2 \wedge dx^3$;
- $d\Omega = \sqrt{|g|}d^4x$, $dV = \dfrac{\sqrt{|g|}}{\sqrt{g_{00}}}d^3x$ – the invariant Riemannian volume element on spacetime and on the spatial manifold respectively.

2.1 Distances, volumes, divergence, codifferential

The (squared) arclength element $ds^2 = g_{ij}dx^i dx^j$ on the spacetime manifold M gives rise to a spatial arclength element, [3], defined as

$$dl^2 = \gamma_{\alpha\beta}dx^\alpha dx^\beta, \qquad \gamma_{\alpha\beta} = -g_{\alpha\beta} + \frac{g_{0\alpha}g_{0\beta}}{g_{00}}, \qquad \alpha, \beta = \overline{1,3}.$$

The determinant of the spacetime metric g is

$$g = -g_{00}\gamma;$$

we only consider reference frames for which $\gamma := \det(\gamma_{ij}) > 0$ and $g_{00} > 0$.

The divergence $div(V) = \dfrac{1}{\sqrt{|g|}}\left(V^i\sqrt{|g|}\right)_{,i}$ of a vector field is written in terms of covariant derivatives as

$$div(V) = V^i{}_{|i}.$$

The generalization of divergence, for differential forms, is the notion of codifferential.

The codifferential of a p-form $\xi = \dfrac{1}{p!}\xi_{i_1 i_2 \dots i_p}dx^{i_1} \wedge \dots \wedge dx^{i_p}$ is the $(p-1)$–form given by $\langle \eta, \delta\xi \rangle = \langle d\eta, \xi \rangle$, where $\langle\ ,\ \rangle$ denotes the inner product of p-forms[1]. For a 2-form, we have

$$(\delta\xi)^i = \xi^{ij}{}_{|j}.$$

2.2 4-potential and electromagnetic tensor

The 4-potential is described in classical general relativity as a 1-form on M :

$$A = A_i(x)dx^i. \tag{1}$$

The electromagnetic tensor (or *Faraday 2-form*) is described as the 2-form

$$F = dA. \tag{2}$$

In local coordinates, this is

$$F = \frac{1}{2}F_{jk}dx^j \wedge dx^k, \tag{3}$$

where

$$F_{jk} = A_{k|j} - A_{j|k}. \tag{4}$$

[1]The inner product of two p-forms $\theta = \frac{1}{p!}\theta_{i_1 \dots i_p}e^{i_1} \wedge \dots \wedge e^{i_p}$ and $\psi = \frac{1}{p!}\psi_{j_1 \dots j_p}e^{j_1} \wedge \dots \wedge e^{j_p}$ is given by $< \theta, \psi >= \frac{1}{p!}\int g^{i_1 j_1} \dots g^{i_p j_p}\theta_{i_1 \dots i_p}\psi_{j_1 \dots j_p}d\Omega$, where the integral is taken on the whole manifold (assuming that the integrands have compact support).

In terms of differential forms, the *homogeneous Maxwell equations*

$$F_{ij|k} + F_{ki|j} + F_{jk|i} = 0. \tag{5}$$

become

$$dF = 0. \tag{6}$$

If we consider as a fundamental object the electromagnetic tensor F, then the homogeneous Maxwell equation $dF = 0$, implies (on a topologically "nice enough" manifold) the existence of a 1-form A, such that $F = dA$. Conversely, if one considers the potential 1-form A as a priori given and *define* F as its exterior differential, then homogeneous Maxwell equation is obtained as an identity. Actually, for a Lagrangian theory of electromagnetism, it is essential to have both a 1-form A and a 2-form F, related by (6).

An important property of the electromagnetic field is *gauge invariance*. Namely, the field strength tensor F is invariant to transformations

$$A \mapsto A + d\psi,$$

where $\psi : M \to \mathbb{R}$ is a differentiable function.

2.3 Lagrangian, equations of motion and inhomogeneous Maxwell equations

The total action attached to the field and to a system of particles is

$$S = \underbrace{-\sum mc \int ds}_{S_p} - \underbrace{\sum \frac{q}{c} \int A_k(x) dx^k}_{S_{int}} - \underbrace{\frac{1}{16\pi c} \int F_{ij} F^{ij} d\Omega}_{S_f}, \tag{7}$$

where m denotes the mass of a particle, q, its charge, c, the speed of light in vacuum and the sums are taken over the particles in the system. The volume integral is taken over a bounded interval of time and over the whole spatial manifold, under the assumption that far away from sources, the field vanishes. Thus, we can actually think the integral as taken over a "large enough" compact domain in M.

Inhomogeneous Maxwell equations are obtained by varying the electromagnetic potential A in the action S (actually, in $S_{int} + S_f$, since S_p does not depend on A). To this aim, one writes $S_{int} + S_f$ as a single volume integral, by means of the notion of charge density ρ.

The integral of ρ over a certain spatial volume provides the total charge situated inside that region:

$$q = \int \rho dV, \tag{8}$$

By using relation (8), S_{int} is written as

$$S_{int} = -\frac{1}{c} \int A_i J^i d\Omega,$$

where

$$J^i := \frac{\rho c}{\sqrt{g_{00}}} \frac{dx^i}{dx^0} \tag{9}$$

define the *4-current* vector field J. Thus, the sum $S_1 := S_{int} + S_f$ becomes

$$S_1 = -\int \left(\frac{1}{c} A_i J^i + \frac{1}{16\pi c} F_{ij} F^{ij} \right) \sqrt{|g|} d^4x.$$

The *inhomogeneous Maxwell equations* are obtained as

$$F^{ij}{}_{|j} = -\frac{4\pi}{c} J^i, \tag{10}$$

which is, actually,

$$(\delta F)^\sharp = -\frac{4\pi}{c} J. \tag{11}$$

The 4-current J identically satisfies the *continuity equation:*

$$div(J) = 0. \tag{12}$$

From a physical point of view, the continuity equation is equivalent to the charge conservation law.

The trajectory of a particle subject to a (fixed) electromagnetic field is determined by varying the action S with respect to the trajectory. That is, we actually have to vary the action $S_2 := S_p + S_{int}$ (written for a single particle). Since the integral S_2 does not depend on the choice of the parameter on the path of integration, we can choose this parameter according to our wish. Choosing the arclength s as a parameter, we have

$$S_2 = -\int \left(mc\sqrt{g_{ij}\dot{x}^i\dot{x}^j} + A_i\dot{x}^i \right) ds. \tag{13}$$

The Euler-Lagrange equations for the above Lagrangian are

$$\frac{D\dot{x}^i}{ds} = \frac{q}{c} F^i{}_j \dot{x}^j, \qquad\qquad i = \overline{0,3}, \tag{14}$$

where $\frac{D\dot{x}^i}{ds} = \frac{d\dot{x}^i}{ds} + \gamma^i{}_{jk}\dot{x}^j\dot{x}^k$. The right hand sides of the above equations provide the expression of the Lorentz force.

2.4 Energy-momentum tensor

In the Minkowski space $(\mathbb{R}^4, \eta)$ $\big(\eta = diag(-1, +1, +1, +1)\big)$ of special relativity, it makes sense to speak about spacetime translations

$$x \mapsto x + a \qquad (a \text{ - constant 4-vector}).$$

Lagrangians in (7) are all invariant with respect to these translations. According to Noether's theorem, the invariance of an action

$$S = \frac{1}{c} \int \Lambda \left(q_{(l)}, \frac{\partial q_{(l)}}{\partial x^i} \right) d\Omega,$$

(for a closed system) to translations implies that the quantities $\tilde{T}^k_i = q_{(l),i} \frac{\partial \Lambda}{\partial q_{(l),k}} - \delta^k_i \Lambda$ are conserved ($div\tilde{T} = 0$). They define a tensor of rank two (the *Noether current* attached to the Lagrangian). The Noether current is generally not symmetric. It can be symmetrized by adding a certain divergence term[2].

The *energy-momentum tensor of the electromagnetic field* in flat space is defined, [3], as the symmetrized Noether current T given by the invariance of the action S_f to spacetime translations.

[2]We assume, as usually, that on the boundary of the integration domain, the involved functions vanish, hence adding divergence terms does not affect the action

For electromagnetism, we have $q_{(k)} = A_{(k)}$ and $\Lambda = -\frac{1}{16\pi}F_{ij}F^{ij}$. One gets the energy-momentum tensor:

$$T^l{}_i = \frac{1}{4\pi}\left(-F^{lk}F_{ik} + \frac{1}{4}\delta^l_i F_{jk}F^{jk}\right). \tag{15}$$

In vacuum (where $J = 0$), the divergence of T vanishes. In the presence of charged matter, by using Maxwell equations, it is proven that the energy-momentum tensor satisfies the identities:

$$T^j{}_{i;j} = -\frac{1}{c}F_{ij}J^j, \tag{16}$$

i.e.:

$$div(T) = -\frac{1}{c}i_J F. \tag{17}$$

The quantity $\frac{1}{c}i_J F$ is called the *density of Lorentz force* and the above relation expresses the conservation of the total energy and momentum of the system.

Conclusion. In a geometric language, the above fundamental equations of electromagnetic field theory can be written briefly as:

- $F = dA$.
- $dF = 0$, $(\delta F)^\sharp = -\frac{4\pi}{c}J$ – Maxwell equations;
- $div(J) = 0$ – continuity equation;
- $div(T) = -\frac{1}{c}i_J F$ – energy-momentum conservation.

3 Finsler spaces with weak metrics

Let, now, $M = \mathbb{R}^4$ be regarded as spacetime manifold. We denote by $(x^i, y^i)_{i=\overline{0,3}}$ the coordinates in a local chart on the tangent bundle TM; the base coordinates x^i play the role of positional variables and the fiber ones y^i, the role of directional variables. We preserve the notations in the previous section, with the only difference that instead of Levi-Civita covariant derivatives, we will use another covariant derivation law. Also, we denote partial derivation with respect to y^i by a dot: $_{\cdot i}$.

Let us suppose that on M we have a Finslerian fundamental function[3] $\mathcal{F} : TM \to \mathbb{R}$ and the corresponding metric tensor:

$$g_{ij}(x, y) = \frac{1}{2}\frac{\partial^2 \mathcal{F}^2}{\partial y^i \partial y^j}.$$

In the following, we suppose that:

- the metric tensor g_{ij} has $(-, +, +, +)$ signature. Actually, it would be rigorous to call the space in this case, a pseudo-Finslerian one. Still, since a lot of authors already use the term "Finsler" for spaces whose metric is not necessarily positive definite, we will adopt this simpler terminology.

- the metric tensor is obtained by a small perturbation of a locally Minkowski metric $\eta_{ij} = \eta_{ij}(y)$:

$$g_{ij} = \eta_{ij}(y) + \varepsilon_{ij}(x, y),$$

(where quadratic terms in ε_{ij} and its derivatives vanish);

[3] We use the notation $\mathcal{F}$ in calligraphic fonts in order to avoid confusion with the electromagnetic 2-form F.

- coordinate changes $(x^i) \longmapsto (x^{i'})$ are such that in any of the considered coordinate systems, η_{ij} do not depend on x (for instance, if $\eta_{ij} = diag(-1,1,1,1)$ is the Minkowski metric of classical special relativity, Lorentz transformations have this property). Thus, in the traditional Finsler-Lagrange geometry approach, we can choose as nonlinear connection the trivial one $N^i{}_j = 0$.

The element of arc length along a curve $t \mapsto x(t)$ is $ds = \mathcal{F}\left(x, \dfrac{dx}{dt}\right) dt$. The spatial metric tensor is defined similarly to the pseudo-Riemannian case: $\gamma_{\alpha\beta} = -g_{\alpha\beta} + \dfrac{g_{0\alpha} g_{0\beta}}{g_{00}}$, $\alpha, \beta \in \{1,2,3\}$ and its determinant is $\det(\gamma_{\alpha\beta}) = \dfrac{\sqrt{|g|}}{\sqrt{g_{00}}}$.

The natural basis on TM is $\left(\partial_i := \dfrac{\partial}{\partial x^i}, \quad \partial_{\bar{\imath}} := \dfrac{\partial}{\partial y^{\bar{\imath}}}\right)$ and its dual basis is $(dx^i, dy^{\bar{\imath}})$. In the following, whenever needed to make a clear distinction, we will denote by $i, j, k, ...$ indices corresponding to base coordinates x^i and by $\bar{\imath}, \bar{\jmath}, \bar{k}, ...$ indices corresponding to fiber ones $y^{\bar{\imath}}$[4]. By capital letters $A, B, C, ...$ we denote indices which run over all values: $A, B, C, ... \in \{i, j, k, ..., \bar{\imath}, \bar{\jmath}, \bar{k}, ...\}$

In order to speak about volumes, divergence and codifferential for objects defined on TM, we need a metric structure on the total space of TM. Hence, we complete g up to metric (an *hv-metric,* [4]) on TM :

$$G_{AB}(x,y) = g_{ij}(x,y) dx^i \otimes dx^j + v_{\bar{\imath}\bar{\jmath}} dy^{\bar{\imath}} \otimes dy^{\bar{\jmath}}. \tag{18}$$

where v is the Euclidean metric[5]. Thus, (TM, G) becomes a pseudo-Riemannian space and we can speak about the Riemannian (invariant) volume element on TM :

$$d\Omega = \sqrt{|G|} d^4x \wedge d^4y.$$

where $G = \det(G_{AB})$. We have, obviously: $G = g \cdot v$, where $g = \det(g_{ij})$, $\quad v = \det(v_{\bar{\imath}\bar{\jmath}})$.

The volume element $d\Omega$ defines a volume element $d\Omega_M$ on M by:

$$d\Omega_M = \sigma(x) d^4x, \qquad \sigma(x) = \int\limits_{D_x} \sqrt{|G|} d^4x \wedge d^4y,$$

where $D_x = \{y \in T_xM \mid v_{\bar{\imath}\bar{\jmath}} y^{\bar{\imath}} y^{\bar{\jmath}} \leq r^2\}$ and $r = \sqrt[4]{2/\pi^2}$ is chosen such that the 3-sphere of radius r in the 4-dimensional Euclidean space has the volume equal to 1, [13]. This volume element generalizes the idea of Holmes-Thompson volume in [9][6]. Integration over y intuitively means "scanning" all the directions which fill a certain neighborhood (a Euclidean sphere) around a point x of spacetime. For a function $f = f(x)$ on M, its integral on a domain $\Delta \subset M$ is

$$\int\limits_{\Delta} f(x) d\Omega_M = \int\limits_{\Delta} f(x) \left(\int\limits_{D_x} \sqrt{|G|} d^4y \right) d^4x = \int f(x) d\Omega.$$

We assume that far away from sources, the field is negligible and the considered time interval is a bounded one; thus, we can consider integrals of a "large enough" compact domain in TM.

[4]Still, since vectors $y \in T_xM$ can be regarded both as tangent vectors $y \sim dx$ to the base manifold and as elements of the fiber, depending on the context, we will use both notations y^i and $y^{\bar{\imath}}$ for their coordinates.

[5]For instance, the manifold topology of Minkowski spacetime is the Euclidean one, hence, in this case, the metric v describes the topological properties of spacetime.

[6]The classical idea of Holmes-Thompson volume involves integration with respect to y on the indicatrix $I_x = \{y \in T_xM |\ g_{ij} y^i y^j = 1\}$. If the Finsler metric g is not positive definite, as in our case, the indicatrix I_x is generally non-compact, hence this classical idea cannot be applied as such. Choosing as vertical part v a positive definite one and integrating on balls given by the metric v solves this problem.

It is convenient to express the results in terms of the following covariant derivation law D:

$$D_{\partial_k}\partial_j = \gamma^i{}_{jk}\partial_i, \quad D_{\partial_k}\partial_{\bar{j}} = 0, \quad D_{\partial_{\bar{k}}}\partial_j = 0, \quad D_{\partial_{\bar{k}}}\partial_{\bar{j}} = 0. \tag{19}$$

where $\gamma^i{}_{jk} = \frac{1}{2}g^{ih}(g_{hj,k} + g_{hk,j} - g_{jk,h})$ are the Christoffel symbols of $g = g(x,y)$. For the components of a vector field $X = X^i\partial_i + X^{\bar{\imath}}\partial_{\bar{\imath}}$ on TM, we will have:

$$X^j{}_{|i} = X^j{}_{,i} + \gamma^j{}_{hi}X^h, \quad X^j{}_{\cdot\bar{\imath}} = \frac{\partial X^j}{\partial y^{\bar{\imath}}}, \quad X^{\bar{j}}{}_{|i} = X^{\bar{j}}{}_{,i}, \quad X^{\bar{j}}{}_{\cdot\bar{\imath}} = \frac{\partial X^{\bar{j}}}{\partial y^{\bar{\imath}}}$$

(where $_{|i}$ denotes covariant derivation by ∂_i and $_{\cdot i}$ means covariant derivation by $\partial_{\bar{\imath}}$). This linear connection is h-metrical, i.e., $g_{ij|k} = 0$, $v_{\bar{\imath}\bar{j}|k} = 0$. Another important notion for a Finsler space is the *Cartan tensor* C given by

$$C^i{}_{j\bar{k}} = \frac{1}{2}g^{ih}g_{hj\cdot\bar{k}}.$$

There hold the relations: $\frac{\partial}{\partial x^i}\left(\ln\sqrt{|g|}\right) = \gamma^i{}_{ji}$ and $\frac{\partial}{\partial y^i}\left(\ln\sqrt{|g|}\right) = C^i{}_{ji}$, hence the divergence of a vector field $V = V^i\partial_i + V^{\bar{\imath}}\partial_{\bar{\imath}}$ on TM can be written as:

$$div(V) = \frac{1}{\sqrt{|g|}}\left[\left(V^i\sqrt{|g|}\right)_{,i} + \left(V^{\bar{\imath}}\sqrt{|g|}\right)_{\cdot\bar{\imath}}\right] = V^i{}_{|i} + V^{\bar{\imath}}{}_{\cdot\bar{\imath}} + V^{\bar{\imath}}C^h{}_{h\bar{\imath}}. \tag{20}$$

The codifferential $\delta\xi$ of a 2-form $\delta\xi = \omega_i dx^i + \omega_{\bar{\imath}}dy^{\bar{\imath}}$ is locally given by

$$(\delta\xi)^i = \xi^{ij}{}_{|j} + \xi^{i\bar{j}}{}_{\cdot\bar{j}} + \xi^{i\bar{j}}C^l{}_{l\bar{j}}, \quad (\delta\xi)^{\bar{\imath}} = \xi^{\bar{\imath}j}{}_{|j} + \xi^{\bar{\imath}\bar{j}}{}_{\cdot\bar{j}} + \xi^{\bar{\imath}\bar{j}}C^l{}_{l\bar{j}}.$$

4 Faraday 2-form, homogeneous Maxwell equations

As shown in [1], since the inhomogeneous Maxwell equations involve the metric tensor g, in the case when $g_{ij} = g_{ij}(x,y)$, their solutions would generally depend on y. Thus, we will consider the following 1-form on TM :

$$A = A_i(x,y)dx^i. \tag{21}$$

The only restriction we will impose to A is that

$$A_i(x,\lambda y) = A_i(x,y),$$

i.e., we will allow A to depend on the direction of y, but not on its magnitude[7].

In [1], we defined the generalized Faraday 2-form (the electromagnetic tensor) as:

$$F = dA; \tag{22}$$

in local coordinates, this is

$$F := \frac{1}{2}F_{ij}dx^i \wedge dx^j + F_{i\bar{j}}dx^i \wedge dy^{\bar{j}}, \tag{23}$$

where

$$F_{ij} = A_{j|i} - A_{i|j}, \quad F_{i\bar{j}} = -A_{i\cdot\bar{j}}. \tag{24}$$

[7]In [1] and [11], we imposed a supplementary restriction, namely, that $A_{i\cdot k}y^i = 0$. Here, this condition will be treated as a "gauge" and not as a part of the definition.

In particular, if $A = A(x)$ does not depend on the directional variables, we get $F = \frac{1}{2}(A_{j|i} - A_{i|j})dx^i \wedge dx^j$.

The electromagnetic tensor F remains invariant under transformations

$$A(x,y) \ \mapsto A(x,y) + d\lambda(x), \tag{25}$$

where $\lambda : M \to \mathbb{R}$ is a scalar function, since $d(A + d\lambda) = dA + d(d\lambda) = dA$.

If F is defined as above, then $dF = ddA = 0$. In other words, we get as an identity the generalized *homogeneous Maxwell equation*:

$$dF = 0. \tag{26}$$

In local coordinates, equation (26) is read as:

$$F_{ij|k} + F_{ki|j} + F_{jk|i} = 0; \quad F_{\bar{\imath}j|k} + F_{k\bar{\imath}|j} + F_{jk\cdot\bar{\imath}} = 0, \qquad F_{k\bar{\imath}\cdot\bar{\jmath}} + F_{\bar{\jmath}k\cdot\bar{\imath}} = 0.$$

The first set in the above is the analogue of the usual homogeneous Maxwell equations. The two other sets appear due to the y-dependence of A.

In the above, we have started from A as an *a priori* given object and defined F as its exterior derivative. Conversely, if we suppose that F is a 2-form as in (23), satisfying $dF = 0$, then there exists, [11], a horizontal form $A = A_i(x,y)dx^i$ such that $F = dA$.

5 Inhomogeneous Maxwell equations

The second term of the total action becomes

$$S_{int} = -\sum \frac{q}{c} \int A_i(x, \dot{x})dx^i.$$

Since A_i are functions on TM, it is natural to transform this integral into an integral on a domain in TM. This can be achieved by writing total charge as an integral:

$$q = \int \frac{\rho(x)}{\sqrt{g_{00}}} \sqrt{G} d^3x \wedge d^4y.$$

With the notation

$$J^i = \frac{\rho c}{\sqrt{g_{00}}} \frac{dx^i}{dx^0}, \tag{27}$$

the integral $\int A_k dx^k$ is written as a volume integral, in a similar way to the Riemannian case (just, here, the integral is on a domain in TM):

$$-\frac{q}{c} \int A_k dx^k = -\frac{1}{c} \int A_i J^i d\Omega. \tag{28}$$

By varying with respect to the potential A the action

$$S_1 = -\frac{1}{c} \int \Big(A_i J^i + \frac{1}{16\pi} F_{AB} F^{AB} \Big) d\Omega. \tag{29}$$

we obtained, [1, 11]:

$$F^{ij}{}_{|j} + F^{i\bar{\jmath}}{}_{\cdot\bar{\jmath}} + F^{i\bar{\jmath}} C^h{}_{h\bar{\jmath}} = -\frac{4\pi}{c} J^i. \tag{30}$$

Notes: 1) In the integral above, in order to make sure that the expression has physical sense, we might need to adjust measurement units so as to have $[F_{ij}] = [F_{i\bar{\jmath}}]$. This can be done by considering the fiber coordinates $y^{\bar{\imath}}$ as having the same measurement units as the base ones (eventually, by multiplying them by a constant, [11]).

2) We notice a certain resemblance between the term $F^{i\bar{j}}{}_{\cdot\bar{j}} + F^{i\bar{j}}C^{h}{}_{h\bar{j}} = \left(F^{i\bar{j}}\sqrt{|g|}\right)_{\cdot\bar{j}}$ and the idea of bound current in a material medium.

Equations (30) gave the idea to formally generalize the *inhomogeneous Maxwell equation* as

$$(\delta F)^{\sharp} = -\frac{4\pi}{c}J. \tag{31}$$

In local coordinates, this is:

$$\begin{aligned} F^{ij}{}_{|j} + F^{i\bar{j}}{}_{\cdot\bar{j}} + F^{i\bar{j}}C^{l}{}_{l\bar{j}} &= -\frac{4\pi}{c}J^{i}, \\ F^{\bar{\imath}j}{}_{|j} &= -\frac{4\pi}{c}J^{\bar{\imath}}, \end{aligned} \tag{32}$$

We notice, in comparison to the equations in [1], the appearance of an extra set of equations and of the quantities $J^{\bar{\imath}}$ which are "coupled" on TM to the usual components of the 4-current J^{i}. Thus, we obtained a vector field

$$J = J^{i}\partial_{i} + J^{\bar{\imath}}\partial_{\bar{\imath}}$$

on TM, whose horizontal component $J^{i}\partial_{i}$ is the usual 4-current (plus the correction due to anisotropy). In the following, we will see that the new component $J^{\bar{\imath}}\partial_{\bar{\imath}}$ plays an important role in the continuity equation and in the Finslerian analogue of energy-momentum conservation law.

Remark. In particular, if $A_i = A_i(x)$, then $J^{\bar{\imath}} = 0$.

6 Continuity equation and gauge invariance

Above, we have seen that

$$-\frac{4\pi}{c}J_{\flat} = \delta F. \tag{33}$$

It immediately follows: $-\dfrac{4\pi}{c}\delta J_{\flat} = \delta\delta F = 0$[8], which is, $div(J) = 0$. In other words:

Proposition 1. *There identically holds the generalized continuity equation:*

$$div(J) = 0. \tag{34}$$

The invariance of the electromagnetic tensor F to transformations $A(x,y) \mapsto \tilde{A}(x,y) := A(x,y) + d\lambda(x)$ of the 4-potential implies the invariance of the first term S_p and the third one S_f in the general action (7). The continuity equation (34) insures that $\tilde{S}_{int} = -\int \tilde{A}_i J^i d\Omega$ equals S_{int} plus a boundary term. Hence, these transformations do not affect the action (7).

7 Equations of motion of charged particles

Equations of motion of charged particles are obtained by varying the trajectory $x = x(t)$ in the first two terms of (7):

$$S_2 = -\int \left(mc\sqrt{g_{ij}(x,\dot{x})\dot{x}^i\dot{x}^j} + \frac{q}{c}A_k(x,\dot{x})\dot{x}^k \right) dt. \tag{35}$$

[8] We have used the identity $\delta\delta\omega = 0$.

The condition $A(x, \lambda y) = A(x, y)$ insures that the action S_2 is invariant under eventual changes of parameter $t \mapsto t'$ of the curve, hence we are free to choose this parameter.

A further restriction can be imposed on the y-dependence of A in order to make the approach more elegant and provide a simple expression for the equations of motion of charged particles[9].

In *isotropic* (pseudo-Riemannian) spaces, if we assume that $A = A(x)$, then there exists only one potential providing a given interaction Lagrangian $L_{int} = A_i(x)y^i$. But in anisotropic spaces, where $A_i = A_i(x, y)$, a given Lagrangian $L_{int} = A_i(x, y)y^i$ can be given by infinitely many functions $A_i = A_i(x, y)$. Thus, to a Lagrangian L_{int}, it corresponds a whole equivalence class of potentials A. It appears as convenient to choose from each class the representative for which

$$A_{k\cdot i}y^k = 0. \tag{36}$$

We will call this condition upon A, the *gradient gauge.* The gradient gauge is equivalent to the condition: $A_i = \dfrac{\partial(A_k y^k)}{\partial y^i}$.

Under the above assumption, the Euler-Lagrange equations attached to S_2 (interpreted as equations of motion of charged particles in Finsler spaces) are:

$$mc\frac{Dy^i}{ds} = \frac{q}{c}F^i{}_j y^j + \frac{q}{c}F^i{}_{\bar{j}}\frac{dy^{\bar{j}}}{ds}, \quad y^i = \frac{dx^i}{ds}, \tag{37}$$

where $\dfrac{Dy^i}{ds} = \dfrac{dy^i}{ds} + \gamma^i{}_{jk}y^j y^k$.

The first term in the right hand side of (37) is similar to the usual one in pseudo-Riemannian spaces, while the second one $\tilde{F}^i = \dfrac{q}{c}F^i{}_{\bar{j}}\dfrac{dy^{\bar{j}}}{ds}$ appears due to the dependence of A (and actually, of the tensor g_{ij}) on the variable y. Both the "traditional" Lorentz force term and the correction $\tilde{F}^i\partial_i$ are orthogonal to the velocity 4-vector $y = \dot{x}$:

$$g_{ij}F^i y^j = 0, \quad g_{ij}\tilde{F}^i y^j = 0. \tag{38}$$

8 Generalized stress-energy-momentum tensor

Let us suppose that $g_{ij} = \eta_{ij}(y)$. Spacetime translations $\bar{x}^i = x^i + \varepsilon^i$, $i = \overline{0,3}$ induce the following transformation on TM :

$$\bar{x}^i = x^i + \varepsilon^i, \quad \bar{y}^i = y^i. \tag{39}$$

We will call *generalized energy-momentum tensor on TM*, the symmetrized Noether current given by the invariance to transformations (39) of the action S_F.

By Noether's theorem, we get a tensor consisting of two blocks:

$$T = T_{ij}dx^i \otimes dx^j + T_{i\bar{j}}dx^i \otimes dx^{\bar{j}}$$

where:

$$T^l{}_i = \frac{1}{4\pi}\left(-F^{lB}F_{iB} + \frac{1}{4}\delta^l_i F_{BC}F^{BC}\right); \tag{40}$$

$$T^{\bar{l}}{}_i = -\frac{1}{4\pi}F^{\bar{l}k}F_{ik}. \tag{41}$$

The horizontal component $T_{ij}dx^i \otimes dx^j$ is the the usual energy-momentum tensor (plus some correction due to anisotropy), while the mixed one $T_{i\bar{j}}dx^i \otimes dy^{\bar{j}}$ is new.

[9]In [1], this restriction was taken as part of the definition of the potential.

By using Maxwell equations, we get:

$$\frac{1}{\sqrt{|g|}}\left[\frac{\partial}{\partial x^j}\left(T^j{}_i\sqrt{|g|}\right)+\frac{\partial}{\partial y^{\bar{j}}}\left(T^{\bar{j}}{}_i\sqrt{|g|}\right)\right]=-\frac{1}{c}\left(F_{ij}J^j+F_{i\bar{j}}J^{\bar{j}}\right). \tag{42}$$

9 Conclusion

For 4-dimensional pseudo-Finsler spaces $(M,\mathcal{F})$ with metrics obtained as small deformations of locally Minkowskian metrics, we have extended several usual notions of electromagnetic field theory.

The 4-potential is defined as a horizontal 1-form $A = A_i(x,y)dx^i$ on the tangent bundle TM, having its components A_i homogeneous of degree 0 in y. The generalized electromagnetic tensor is the 2-form $F = dA$. Maxwell equations on TM are then written as:

$$dF = 0, \quad (\delta F)^\sharp = -\frac{4\pi}{c}J.$$

The TM-current $J = J^i\partial_i + J^{\bar{\imath}}\partial_{\bar{\imath}}$ is a vector field on TM, satisfying identically $div\ J = 0$. Its horizontal component $J^i\partial_i$ provides the usual 4-current (plus a correction term due to the y-dependence of A), while the vertical one $J^{\bar{\imath}}\partial_{\bar{\imath}}$ is new.

Further, the generalized energy-momentum tensor in flat Finsler spaces is defined as the symmetrized Noether current corresponding to invariance to spacetime translations of the field Lagrangian. We obtained

$$T = T_{ij}dx^i\otimes dx^j + T_{i\bar{j}}dx^i\otimes dy^{\bar{j}}, \tag{43}$$

$$T_{\ iA} = \frac{1}{4\pi}\left(-F_A{}^BF_{iB}+\frac{1}{4}\delta^l_AF_{BC}F^{BC}\right), \tag{44}$$

(where δ^l_j is the Kronecker delta and $\delta^l_{\bar{j}} = 0$). This tensor satisfies the identities:

$$div(T) = -\frac{1}{c}\left(F_{ij}J^j + F_{i\bar{j}}J^{\bar{j}}\right).$$

This approach offers an alternative to the existing one by R. Miron and collaborators, [5, 6, 4].

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УРАВНЕНИЯ ЭЛЕКТРОМАГНЕТИЗМА В НЕКОТОРЫХ АНИЗОТРОПНЫХ ПРОСТРАНСТВАХ СПЕЦИАЛЬНОГО ВИДА. ЧАСТЬ 2

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Используя формализм вариационного исчисления и внешней производной, мы предложили в [1] новый геометрический подход к электромагнетизму в пространствах с метрикой, полученной в результате малых деформаций плоской финслеровой метрики. Эти идеи были распространены на финслеровы пространства общего вида в [11].

В настоящей работе мы рассматриваем более детально вопросы связанные с обобщенными токами, областью интегрирования и калибровочной инвариантностью. Также для плоских финслеровых пространств мы определяем обобщенный тензор энергии-импульса как симметризованный ток соответствующий инвариантности лагранжиана поля по отношению к трансляциям пространства-времени.

Ключевые слова: Финслерово пространство, касательное расслоение, электромагнитный тензор, электромагнитный потенциал, тензор энергии-импульса.