JET LOCAL RIEMANN-FINSLER GEOMETRY FOR THE THREE-DIMENSIONAL TIME

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The aim of this paper is to develop on the 1-jet space the Finsler-like geometry (in the sense of distinguished (d-) connection, d-torsions and d-curvatures) for a 1-parameter deformation of the Berwald-Moór metric of order three. Some field-like geometrical theories (gravitational-like and electromagnetic-like) produced by our 1-parameter deformation of the Berwald-Moór metric are also exposed.

Key Words: 1-parameter deformation of the Berwald-Moór metric of order three, nonlinear connection, Cartan canonical linear connection, d-torsions and d-curvatures, Einstein-like geometrical equations.
MSC(2000): 53C60, 53C80, 83C22.

1 Introduction

It is a well known fact that, in order to create the Relativity Theory, Einstein used Riemannian geometry instead of classical Euclidean geometry, the first one representing a natural mathematical model for local *isotropic* space-time. Although the use of Riemannian geometry was indeed a genial idea, there are recent studies of physicists that suggest a *non-isotropic* perspective of space-time. For example, in Pavlov's opinion [15], the concept of inertial body mass emphasizes the necessity to study the local non-isotropic spaces. Obviously, for the study of non-isotropic physical phenomena, the Finsler geometry is very useful as a mathematical framework.

The studies of Russian scholars (Asanov [1], Garas'ko [4] and Pavlov [5, 14, 15]) emphasize the importance of Finsler geometry which is characterized by the total equality in rights of all non-isotropic directions. For such a reason, Asanov, Pavlov and their co-workers underline the important role played in theory of space-time structure and gravitation (as well as in unified gauge field theories) by Berwald-Moór metric (whose certain Finsler geometrical properties are studied by Matsumoto and Shimada in the works [6, 7, 16])

$$F : TM \to \mathbb{R}, \qquad F(y) = \left(y^1 y^2 ... y^n\right)^{1/n}.$$

Because any of such directions can be related to the proper time of an inertial reference frame, Pavlov considers that it is appropriate as such spaces to be generically called *"multi-dimensional times"* [15]. In the framework of 3- and 4-dimensional linear space with Berwald-Moór metric (i.e. the three- and four-dimensional time), Pavlov and his co-workers [5, 14, 15] offer some new physical approaches and geometrical interpretations such as:

1. physical events = points in multi-dimensional time;
2. straight lines = shortest curves;
3. intervals = distances between the points along of a straight line;
4. simultaneous surfaces = the surfaces of simultaneous physical events.

According to Olver's opinion [13], we consider that the 1-jet fibre bundle is a basic object in the study of classical and quantum field theories. For such geometrical and physical reasons, this paper is devoted to the development on the 1-jet space $J^1(\mathbb{R}, M^3)$ of a Finsler-like geometry

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(together with some gravitational-like and electromagnetic-like geometrical models) for the *t-deformation of the Berwald-Moór metric* given by

$$\mathring{F} : J^1(\mathbb{R}, M^3) \to \mathbb{R}, \qquad \mathring{F}(t,y) = \sqrt{h^{11}(t)} \cdot \sqrt[3]{y_1^1 y_1^2 y_1^3},$$

where $h_{11}(t)$ is a Riemannian metric on $\mathbb{R}$ and $(t, x^1, x^2, x^3, y_1^1, y_1^2, y_1^3)$ are the coordinates of the 1-jet space $J^1(\mathbb{R}, M^3)$.

Remark 1.1. *If we take the particular local Riemannian metric*

$$h_{11}(t) = e^{-2\sigma(t)} > 0,$$

it follows that $\mathring{F}$ becomes a t-conformal deformation of the ***jet Berwald-Moór metric of order three***

$$BM_3(y) = \sqrt[3]{y_1^1 y_1^2 y_1^3}.$$

The differential geometry (in the sense of Cartan linear connections, d-torsions, d-curvatures, gravitational-like and electromagnetic-like geometrical models) produced by an arbitrary jet Lagrangian function

$$L : J^1(\mathbb{R}, M^n) \to \mathbb{R}$$

is now completely done in the second author's paper [12]. We point out that the geometrical ideas from [12] are similar (but however distinct ones) to those exposed by Miron and Anastasiei in the classical Lagrangian geometry on tangent bundle [8]. More accurately, the geometrical ideas from [12] (which we called the jet geometrical theory of *relativistic rheonomic (t-dependent) Lagrange spaces*) were initially stated by Asanov in [2] and developed further in the book [11] by the second author of this paper.

In the sequel, we apply the general geometrical results from [12] to the particular rheonomic (t-deformed) Berwald-Moór metric $\mathring{F}$, in order to obtain what we called the *jet local Riemann-Finsler geometry for three-dimensional time.*

2 Preliminary notations and formulas

Let $(\mathbb{R}, h_{11}(t))$ be a Riemannian manifold, where $\mathbb{R}$ is the set of real numbers. The Christoffel symbol of the Riemannian metric $h_{11}(t)$ is

$$\varkappa_{11}^1 = \frac{h^{11}}{2}\frac{dh_{11}}{dt}, \qquad h^{11} = \frac{1}{h_{11}} > 0.$$

Let also M^3 be a manifold of dimension three, whose local coordinates are (x^1, x^2, x^3). Let us consider the 1-jet space $J^1(\mathbb{R}, M^3)$, whose local coordinates are

$$(t, x^1, x^2, x^3, y_1^1, y_1^2, y_1^3).$$

These transform by the rules (the Einstein convention of summation is used throughout this work):

$$\widetilde{t} = \widetilde{t}(t), \quad \widetilde{x}^p = \widetilde{x}^p(x^q), \quad \widetilde{y}_1^p = \frac{\partial \widetilde{x}^p}{\partial x^q}\frac{dt}{d\widetilde{t}} \cdot y_1^q, \qquad p, q = \overline{1,3}, \tag{2.1}$$

where $d\widetilde{t}/dt \neq 0$ and rank $(\partial \widetilde{x}^p / \partial x^q) = 3$. We consider that the manifold M^3 is endowed with a tensor of rank $(0,3)$, given by local components $G_{pqr}(x)$. This is totally symmetric in the indices p, q and r. Suppose that the d-tensor

$$G_{ij1} = 6G_{ijp}y_1^p,$$

is non-degenerate, that is there exists the d-tensor G^{jk1} on $J^1(\mathbb{R}, M^3)$ such that $G_{ij1}G^{jk1} = \delta_i^k$.

In this geometrical context, if we use the notation $G_{111} = G_{pqr}y_1^p y_1^q y_1^r$, we can consider the *third-root Finsler-like function* [16], [3] (this is 1-positive homogenous in the variable y):

$$F(t, x, y) = \sqrt[3]{G_{pqr}(x)y_1^p y_1^q y_1^r} \cdot \sqrt{h^{11}(t)} = \sqrt[3]{G_{111}(x, y)} \cdot \sqrt{h^{11}(t)}, \tag{2.2}$$

where the Finsler function F has as domain of definition all values (t, x, y) which verify the condition $G_{111}(x, y) \neq 0$ (i.e. the domain where we can y-differentiate the function $F(t, x, y)$).

If we denote $G_{i11} = 3G_{ipq}y_1^p y_1^q$, then the 3-positive homogeneity of the "y-function" G_{111} (this is in fact a d-tensor on the 1-jet space $J^1(\mathbb{R}, M^3)$) leads to equalities:

$$G_{i11} = \frac{\partial G_{111}}{\partial y_1^i}, \quad G_{i11}y_1^i = 3G_{111}, \quad G_{ij1}y_1^j = 2G_{i11},$$

$$G_{ij1} = \frac{\partial G_{i11}}{\partial y_1^j} = \frac{\partial^2 G_{111}}{\partial y_1^i \partial y_1^j}, \quad G_{ij1}y_1^i y_1^j = 6G_{111}, \quad \frac{\partial G_{ij1}}{\partial y_1^k} = 6G_{ijk}.$$

The *fundamental metrical d-tensor* produced by F is given by formula

$$g_{ij}(t, x, y) = \frac{h_{11}(t)}{2} \frac{\partial^2 F^2}{\partial y_1^i \partial y_1^j}.$$

By direct computations, the fundamental metrical d-tensor takes the form

$$g_{ij}(x, y) = \frac{G_{111}^{-1/3}}{3} \left[G_{ij1} - \frac{1}{3G_{111}} G_{i11}G_{j11} \right]. \tag{2.3}$$

Moreover, taking into account that the d-tensor G_{ij1} is non-degenerate, we deduce that the matrix $g = (g_{ij})$ admits the inverse $g^{-1} = (g^{jk})$. The entries of the inverse matrix g^{-1} are given by

$$g^{jk} = 3G_{111}^{1/3} \left[G^{jk1} + \frac{G_1^j G_1^k}{3\left(G_{111} - \mathcal{G}_{111}\right)} \right], \tag{2.4}$$

where $G_1^j = G^{jp1}G_{p11}$ and $3\mathcal{G}_{111} = G^{pq1}G_{p11}G_{q11}$.

3 t-Deformation of the Berwald-Moór metric

Starting from this Section, we will focus only on the *t-deformation of the Berwald-Moór metric of order three* which is the Finsler-like metric (2.2) for particular case

$$G_{pqr} = \begin{cases} \dfrac{1}{3!}, & \{p, q, r\} \text{ - distinct indices} \\ 0, & \text{otherwise.} \end{cases}$$

Consequently, the t-deformation of the Berwald-Moór metric of order three is given by

$$\mathring{F}(t, y) = \sqrt{h^{11}(t)} \cdot \sqrt[3]{y_1^1 y_1^2 y_1^3}. \tag{3.1}$$

Moreover, using preceding notations and formulas, we obtain the following relations:

$$G_{111} = y_1^1 y_1^2 y_1^3, \quad G_{i11} = \frac{G_{111}}{y_1^i},$$

$$G_{ij1} = (1-\delta_{ij})\,\frac{G_{111}}{y_1^i y_1^j}\ \text{(no sum by } i \text{ or } j),$$

where δ_{ij} is the Kronecker symbol. Because we have

$$\det\left(G_{ij1}\right)_{i,j=\overline{1,3}} = 2G_{111} \neq 0,$$

we find

$$G^{jk1} = \frac{(1-2\delta^{jk})}{2G_{111}} y_1^j y_1^k\ \text{(no sum by } j \text{ or } k).$$

It follows that we have $\mathcal{G}_{111} = (1/2)G_{111}$ and $G_1^j = (1/2)y_1^j$.

If we replace the preceding computed entities into formulas (2.3) and (2.4), we get

$$g_{ij} = \frac{(2-3\delta_{ij})}{9}\frac{G_{111}^{2/3}}{y_1^i y_1^j}\ \text{(no sum by } i \text{ or } j) \tag{3.2}$$

and

$$g^{jk} = (2-3\delta^{jk})G_{111}^{-2/3} y_1^j y_1^k\ \text{(no sum by } j \text{ or } k). \tag{3.3}$$

Using a general formula from paper [12], we find the following geometrical result:

Proposition 3.1. *For the t-deformed Berwald-Moór metric (3.1), the energy action functional*

$$\mathring{\mathbb{E}}(t,x(t)) = \int_a^b \mathring{F}^2(t,y)\sqrt{h_{11}}dt = \int_a^b \sqrt[3]{\{y_1^1 y_1^2 y_1^3\}^2}\cdot h^{11}\sqrt{h_{11}}dt$$

produces on the 1*-jet space* $J^1(\mathbb{R},M^3)$ *the canonical nonlinear connection*

$$\Gamma = \left(M_{(1)1}^{(i)} = -\varkappa_{11}^1 y_1^i,\ N_{(1)j}^{(i)} = 0\right). \tag{3.4}$$

Proof. The Euler-Lagrange equations of the energy action functional $\mathring{\mathbb{E}}$ are equivalent with the equations

$$\frac{d^2x^i}{dt^2} + 2H_{(1)1}^{(i)}\left(t,x^k,y_1^k\right) + 2G_{(1)1}^{(i)}\left(t,x^k,y_1^k\right) = 0, \qquad y_1^k = \frac{dx^k}{dt}, \tag{3.5}$$

where the local geometrical components

$$H_{(1)1}^{(i)} \overset{def}{=} -\frac{1}{2}\varkappa_{11}^1(t) y_1^i$$

and

$$G_{(1)1}^{(i)} \overset{def}{=} \frac{h_{11}g^{ik}}{4}\left[\frac{\partial^2\mathring{F}^2}{\partial x^j\partial y_1^k}y_1^j - \frac{\partial\mathring{F}^2}{\partial x^k} + \frac{\partial^2\mathring{F}^2}{\partial t\partial y_1^k} + \right.$$
$$\left. + \frac{\partial\mathring{F}^2}{\partial y_1^k}\varkappa_{11}^1(t) + 2h^{11}\varkappa_{11}^1 g_{kl} y_1^l\right] := 0$$

represent a *semispray* on the 1-jet space $J^1(\mathbb{R},M^3)$. This semispray produces the *canonical nonlinear connection* (for more details, see the papers [10], [12])

$$\Gamma = \left(M_{(1)1}^{(i)} = 2H_{(1)1}^{(i)} = -\varkappa_{11}^1 y_1^i,\ N_{(1)j}^{(i)} = \frac{\partial G_{(1)1}^{(i)}}{\partial y_1^j} = 0\right).$$

□

Generally speaking, a nonlinear connection $\Gamma = \left(M^{(i)}_{(1)1},\ N^{(i)}_{(1)j}\right)$ on the 1-jet space $J^1(\mathbb{R}, M^3)$ is used for construction of distinguished vector fields (which have a classical tensorial behaviour)

$$\frac{\delta}{\delta t} = \frac{\partial}{\partial t} - M^{(j)}_{(1)1}\frac{\partial}{\partial y^j_1}, \qquad \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^{(j)}_{(1)i}\frac{\partial}{\partial y^j_1}. \tag{3.6}$$

It is important to note that, in our present Finsler-like geometrization, there are a lot of geometrical local components (such as the components of Cartan linear connection, d-torsions, d-curvatures etc.) whose geometrical construction involves the M-horizontal covariant derivatives $\delta/\delta x^i$. In the case when the nonlinear connection Γ has the components $N^{(i)}_{(1)j}$ equal to zero (see (3.4), for instance), it follows that the M-horizontal covariant derivatives $\delta/\delta x^i$ reduce to the classical partial derivatives $\partial/\partial x^i$. Consequently, the above discussed geometrical local components (e.g., which are dependent only by t and y) vanish in this case. For these reasons, we will use on the 1-jet space $J^1(\mathbb{R}, M^3)$, by an "a priori" definition, the following *non-trivial* local nonlinear connection:

$$\mathring{\Gamma} = \left(M^{(i)}_{(1)1} = -\varkappa^1_{11}y^i_1,\ N^{(i)}_{(1)j} = -\frac{\varkappa^1_{11}}{2}\delta^i_j\right). \tag{3.7}$$

Beside the non-triviality of the components $N^{(i)}_{(1)j}$, we have choosen the nonlinear connection (3.7) such that its attached *harmonic curves* be straight lines (this is because the Euler-Lagrange equations (3.5) also have as solutions only pieces of straight lines). In order to be more clear, we recall that the equations of the harmonic curves of the nonlinear connection (3.7) are given by [10]

$$\frac{d^2x^j}{dt^2} + M^{(j)}_{(1)1}\left(t, x^k(t), \frac{dx^k}{dt}\right) + N^{(j)}_{(1)m}\left(t, x^k(t), \frac{dx^k}{dt}\right)\frac{dx^m}{dt} = 0. \tag{3.8}$$

It follows that the equations (3.8) are equivalent to

$$\frac{d^2x^j}{dt^2} = \frac{3}{4}\frac{1}{h_{11}}\frac{dh_{11}}{dt}\frac{dx^j}{dt}. \tag{3.9}$$

Obviously, the equations (3.9) have the general solution

$$x^j(t) = a^j\int_{t_0}^{t}(h_{11})^{3/4}(\sigma)d\sigma + b^j,$$

where $a^j, b^j \in \mathbb{R}$. In other words, the equations (3.9) have as solutions only pieces of the straight lines

$$\frac{x^1 - b^1}{a^1} = \frac{x^2 - b^2}{a^2} = \frac{x^3 - b^3}{a^3}.$$

Remark 3.2. *We point out that the above terminology of* ***harmonic curves (autoparallel curves*** *in Miron's terminology [8]) comes from the particular form of equations (3.8) for the particular* ***global*** *nonlinear connection*

$$\hat{\Gamma} = \left(\hat{M}^{(j)}_{(1)1} = -\varkappa^1_{11}y^j_1,\ \hat{N}^{(j)}_{(1)i} = \gamma^j_{im}y^m_1\right), \tag{3.10}$$

where $\varkappa^1_{11}(t)$ and $\gamma^i_{jk}(x)$ represent the Christoffel symbols of the Riemannian manifolds $(\mathbb{R}, h_{11}(t))$ and $(M^3, \varphi_{ij}(x))$. It is obvious that, for the particular nonlinear connection (3.10), the equations (3.8) become the ***equations of harmonic maps (curves)***

$$\begin{gathered}\frac{d^2x^i}{dt^2} - \varkappa^1_{11}(t)\frac{dx^i}{dt} + \gamma^i_{jk}(x)\frac{dx^j}{dt}\frac{dx^k}{dt} = 0 \Leftrightarrow \\ h^{11}\left[\frac{d^2x^i}{dt^2} - \varkappa^1_{11}(t)\frac{dx^i}{dt} + \gamma^i_{jk}(x)\frac{dx^j}{dt}\frac{dx^k}{dt}\right] = 0.\end{gathered} \tag{3.11}$$

Remark 3.3. *Note that the components $N^{(i)}_{(1)j}$ of the nonlinear connection (3.7), which are given in the local chart $\mathcal{U}$ by the functions*

$$\mathring{N} = \left(N^{(i)}_{(1)j} = -\frac{\varkappa^1_{11}}{2}\delta^i_j \right),$$

have not a global character on the 1-jet space $J^1(\mathbb{R}, M^3)$, but have only a local character. In conclusion, taking into account the general transformation rules (see [10])

$$\widetilde{N}^{(k)}_{(1)l} = N^{(j)}_{(1)i}\frac{dt}{d\widetilde{t}}\frac{\partial x^i}{\partial \widetilde{x}^l}\frac{\partial \widetilde{x}^k}{\partial x^j} - \frac{\partial x^i}{\partial \widetilde{x}^l}\frac{\partial \widetilde{y}^k_1}{\partial x^i}, \tag{3.12}$$

it follows that $\mathring{N}$ has in the local chart $\widetilde{\mathcal{U}}$ the following components:

$$\widetilde{N}^{(k)}_{(1)l} = -\frac{\widetilde{\varkappa}^1_{11}}{2}\delta^k_l + \frac{1}{2}\frac{d\widetilde{t}}{dt}\frac{d^2 t}{d\widetilde{t}^2}\delta^k_l + \frac{\partial \widetilde{x}^k}{\partial x^m}\frac{\partial^2 x^m}{\partial \widetilde{x}^l \partial \widetilde{x}^r}\widetilde{y}^r_1.$$

4 The Cartan $\mathring{\Gamma}$-linear connection. d-Torsions and d-curvatures

We use the nonlinear connection (3.7) for construction of dual *adapted bases* of d-vector fields

$$\left\{ \frac{\delta}{\delta t} = \frac{\partial}{\partial t} + \varkappa^1_{11} y^p_1 \frac{\partial}{\partial y^p_1}, \ \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} + \frac{\varkappa^1_{11}}{2}\frac{\partial}{\partial y^i_1}, \ \frac{\partial}{\partial y^i_1} \right\} \subset \mathcal{X}(E) \tag{4.1}$$

and d-covector fields

$$\left\{ dt, \ dx^i, \ \delta y^i_1 = dy^i_1 - \varkappa^1_{11} y^i_1 dt - \frac{\varkappa^1_{11}}{2} dx^i \right\} \subset \mathcal{X}^*(E), \tag{4.2}$$

where $E = J^1(\mathbb{R}, M^3)$. Note that, under a change of coordinates (2.1), the elements of adapted bases (4.1) and (4.2) must transform as classical tensors. Consequently, all subsequent geometrical objects on the 1-jet space $J^1(\mathbb{R}, M^3)$ (such as the Cartan canonical $\mathring{\Gamma}$-linear connection, torsion, curvature etc.) will be described in local adapted components.

Using a general result from [12], by direct computations, we can give the following important geometrical result:

Proposition 4.1. *The Cartan canonical $\mathring{\Gamma}$-linear connection, produced by the t-deformed Berwald-Moór metric (3.1), has the following adapted components:*

$$C\mathring{\Gamma} = \left(\varkappa^1_{11}, \ G^k_{j1} = 0, \ L^i_{jk} = \frac{\varkappa^1_{11}}{2} C^{i(1)}_{j(k)}, \ C^{i(1)}_{j(k)} \right),$$

where, if we use the notation

$$A^i_{jk} = \frac{3\delta^i_j + 3\delta^i_k + 3\delta_{jk} - 9\delta^i_j\delta_{jk} - 2}{9} \ \textit{(no sum by } i, \ j \textit{ or } k\textit{)}$$

we have

$$C^{i(1)}_{j(k)} = A^i_{jk} \cdot \frac{y^i_1}{y^j_1 y^k_1} \ \textit{(no sum by } i, \ j \textit{ or } k\textit{)}.$$

Proof. Via the t-deformed Berwald-Moór derivative operators (4.1), we use the general formulas which give the adapted components of the Cartan canonical connection, namely [12]

$$G^k_{j1} = \frac{g^{km}}{2}\frac{\delta g_{mj}}{\delta t}, \quad L^i_{jk} = \frac{g^{im}}{2}\left(\frac{\delta g_{jm}}{\delta x^k} + \frac{\delta g_{km}}{\delta x^j} - \frac{\delta g_{jk}}{\delta x^m} \right),$$

$$C^{i(1)}_{j(k)} = \frac{g^{im}}{2}\left(\frac{\partial g_{jm}}{\partial y^k_1} + \frac{\partial g_{km}}{\partial y^j_1} - \frac{\partial g_{jk}}{\partial y^m_1} \right) = \frac{g^{im}}{2}\frac{\partial g_{jk}}{\partial y^m_1}.$$

Remark 4.2. *The Cartan canonical connection $C\mathring{\Gamma}$ has the metrical properties:*

$$h_{11/1} = h^{11}{}_{/1} = 0, \quad h_{11|k} = h^{11}{}_{|k} = 0, \quad h_{11}|_{(k)}^{(1)} = h^{11}|_{(k)}^{(1)} = 0,$$
$$g_{ij/1} = g^{ij}{}_{/1} = 0, \quad g_{ij|k} = g^{ij}{}_{|k} = 0, \quad g_{ij}|_{(k)}^{(1)} = g^{ij}|_{(k)}^{(1)} = 0,$$

where "$_{/1}$", "$_{|k}$" and "$|_{(k)}^{(1)}$" are the $\mathbb{R}$***-horizontal****, M**-horizontal** and **vertical covariant derivatives** produced by the Cartan $\mathring{\Gamma}$-linear connection $C\mathring{\Gamma}$. For more details upon the local expressions of the above covariant derivatives applied to the components of d-tensors, see paper [12]. Consequently, in our jet Finsler-like geometrization, the Cartan canonical connection plays a similar role to that of Levi-Civita connection in Riemannian spaces.*

Remark 4.3. *The below properties of the vertical d-tensor $C^{i(1)}_{j(k)}$ are true (summation by m):*

$$C^{i(1)}_{j(k)} = C^{i(1)}_{k(j)}, \quad C^{i(1)}_{j(m)} y^m_1 = 0, \quad C^{m(1)}_{j(m)} = 0. \tag{4.3}$$

For similar properties, see also the papers [3], [7], [9] or [16].

Remark 4.4. *The coefficients A^l_{ij} have the following values:*

$$A^l_{ij} = \begin{cases} -\dfrac{2}{9}, & i \neq j \neq l \neq i \\ \dfrac{1}{9}, & i = j \neq l \text{ or } i = l \neq j \text{ or } j = l \neq i \\ -\dfrac{2}{9}, & i = j = l. \end{cases} \tag{4.4}$$

Proposition 4.5. *The Cartan canonical connection $C\mathring{\Gamma}$ of the t-deformation of the Berwald-Moór metric (given by (3.1)) has **three** effective adapted local torsion d-tensors:*

$$P^{(k)\ (1)}_{(1)i(j)} = -\frac{\varkappa^1_{11}}{2} C^{k(1)}_{i(j)}, \quad P^{k(1)}_{i(j)} = C^{k(1)}_{i(j)},$$
$$R^{(k)}_{(1)1j} = \frac{1}{2}\left[\frac{d\varkappa^1_{11}}{dt} - \varkappa^1_{11}\varkappa^1_{11}\right]\delta^k_j.$$

Proof. A general h-normal Γ-linear connection on the 1-jet space $J^1(\mathbb{R}, M^3)$ is characterized by *eight* effective d-tensors of torsion (for more details, see [12]). For our Cartan canonical connection $C\mathring{\Gamma}$ these reduce to the following *three* (the other five cancel):

$$P^{(k)\ (1)}_{(1)i(j)} = \frac{\partial N^{(k)}_{(1)i}}{\partial y^j_1} - L^k_{ji}, \quad R^{(k)}_{(1)1j} = \frac{\delta M^{(k)}_{(1)1}}{\delta x^j} - \frac{\delta N^{(k)}_{(1)j}}{\delta t}, \quad P^{k(1)}_{i(j)} = C^{k(1)}_{i(j)}.$$

□

Proposition 4.6. *The Cartan canonical connection $C\mathring{\Gamma}$ of the t-deformation of the Berwald-Moór metric (given by (3.1)) has **three** effective adapted local curvature d-tensors:*

$$R^l_{ijk} = \frac{\varkappa^1_{11}\varkappa^1_{11}}{4} S^{l(1)(1)}_{i(j)(k)}, \quad P^{l\ (1)}_{ij(k)} = \frac{\varkappa^1_{11}}{2} S^{l(1)(1)}_{i(j)(k)},$$
$$S^{l(1)(1)}_{i(j)(k)} = \frac{\partial C^{l(1)}_{i(j)}}{\partial y^k_1} - \frac{\partial C^{l(1)}_{i(k)}}{\partial y^j_1} + C^{m(1)}_{i(j)} C^{l(1)}_{m(k)} - C^{m(1)}_{i(k)} C^{l(1)}_{m(j)}.$$

Proof. A general h-normal Γ-linear connection on the 1-jet space $J^1(\mathbb{R}, M^3)$ is characterized by *five* effective d-tensors of curvature (for more details, see [12]). For our Cartan canonical connection $C\mathring{\Gamma}$ these reduce to the following *three* (the other two cancel):

$$R^l_{ijk} = \frac{\delta L^l_{ij}}{\delta x^k} - \frac{\delta L^l_{ik}}{\delta x^j} + L^m_{ij}L^l_{mk} - L^m_{ik}L^l_{mj},$$

$$P^{l\ \ (1)}_{ij(k)} = \frac{\partial L^l_{ij}}{\partial y^k_1} - C^{l(1)}_{i(k)|j} + C^{l(1)}_{i(m)}P^{(m)\ \ (1)}_{(1)j(k)},$$

$$S^{l(1)(1)}_{i(j)(k)} = \frac{\partial C^{l(1)}_{i(j)}}{\partial y^k_1} - \frac{\partial C^{l(1)}_{i(k)}}{\partial y^j_1} + C^{m(1)}_{i(j)}C^{l(1)}_{m(k)} - C^{m(1)}_{i(k)}C^{l(1)}_{m(j)},$$

where

$$C^{l(1)}_{i(k)|j} = \frac{\delta C^{l(1)}_{i(k)}}{\delta x^j} + C^{m(1)}_{i(k)}L^l_{mj} - C^{l(1)}_{m(k)}L^m_{ij} - C^{l(1)}_{i(m)}L^m_{kj}.$$

□

Remark 4.7. *The vertical curvature d-tensor* $S^{l(1)(1)}_{i(j)(k)}$ *has the properties:*

$$S^{l(1)(1)}_{i(j)(k)} + S^{l(1)(1)}_{i(k)(j)} = 0,$$

$$S^{l(1)(1)}_{i(j)(j)} = 0 \text{ (no sum by } j\text{)}.$$

Proposition 4.8. *The expressions of the vertical curvature d-tensor* $S^{l(1)(1)}_{i(j)(k)}$ *are given by:*

1. $S^{l(1)(1)}_{i(i)(k)} = -\frac{1}{9}\frac{y^l_1}{(y^i_1)^2 y^k_1}$ *($i \neq k \neq l \neq i$ and no sum by i);*
2. $S^{l(1)(1)}_{i(j)(i)} = \frac{1}{9}\frac{y^l_1}{(y^i_1)^2 y^j_1}$ *($i \neq j \neq l \neq i$ and no sum by i);*
3. $S^{i(1)(1)}_{i(j)(k)} = 0$ *($i \neq j \neq k \neq i$ and no sum by i);*
4. $S^{l(1)(1)}_{i(l)(k)} = \frac{1}{9y^i_1 y^k_1}$ *($i \neq k \neq l \neq i$ and no sum by l);*
5. $S^{l(1)(1)}_{i(j)(l)} = -\frac{1}{9y^i_1 y^j_1}$ *($i \neq j \neq l \neq i$ and no sum by l);*
6. $S^{l(1)(1)}_{i(i)(l)} = \frac{1}{9(y^i_1)^2}$ *($i \neq l$ and no sum by i or l);*
7. $S^{l(1)(1)}_{i(l)(i)} = -\frac{1}{9(y^i_1)^2}$ *($i \neq l$ and no sum by i or l);*
8. $S^{l(1)(1)}_{l(l)(k)} = 0$ *($k \neq l$ and no sum by l);*
9. $S^{l(1)(1)}_{l(j)(l)} = 0$ *($j \neq l$ and no sum by l).*

Proof. For $j \neq k$, the expression of the vertical curvature tensor $S^{l(1)(1)}_{i(j)(k)}$ takes the form (no sum by i, j, k or l, but with sum by m)

$$S^{l(1)(1)}_{i(j)(k)} = \left[\frac{A^l_{ij}\delta^l_k}{y^i_1 y^j_1} - \frac{A^l_{ik}\delta^l_j}{y^i_1 y^k_1}\right] + \left[\frac{A^l_{ik}\delta_{ij}y^l_1}{(y^i_1)^2 y^k_1} - \frac{A^l_{ij}\delta_{ik}y^l_1}{(y^i_1)^2 y^j_1}\right] + $$
$$+\left[A^m_{ij}A^l_{mk} - A^m_{ik}A^l_{mj}\right]\frac{y^l_1}{y^i_1 y^j_1 y^k_1},$$

where the coefficients A^l_{ij} are given by relations (4.4). □

5 t-Deformed field-like geometrical models constructed on 1-jet three-dimensional time

5.1 Gravitational-like geometrical model

From a geometrical point of view, on the 1-jet three-dimensional time, the t-deformed Berwald-Moór metric (3.1) produces the adapted metrical d-tensor

$$\mathbb{G} = h_{11}dt \otimes dt + g_{ij}dx^i \otimes dx^j + h^{11}g_{ij}\delta y^i_1 \otimes \delta y^j_1, \tag{5.1}$$

where g_{ij} is given by (3.2) and δy^i_1 is given by (4.2). This may be regarded as a *"non-isotropic gravitational potential"* (see Miron and Anastasiei [8]). In such a "physical" terminology, the nonlinear connection $\mathring{\Gamma}$ (used in the construction of distinguished 1-forms δy^i_1) prescribes, probably, a kind of *"interaction"* between (t)-, (x)- and (y)-fields (cf. Ikeda, Miron and Anastasiei).

We postulate that the non-isotropic gravitational potential $\mathbb{G}$ is governed by the *Einstein geometrical equations*

$$\text{Ric}\left(C\mathring{\Gamma}\right) - \frac{\text{Sc}\left(C\mathring{\Gamma}\right)}{2}\mathbb{G} = \mathcal{K}\mathcal{T}, \tag{5.2}$$

where $\text{Ric}\left(C\mathring{\Gamma}\right)$ is the *Ricci d-tensor* associated to the Cartan canonical connection $C\mathring{\Gamma}$ (in Riemannian sense and described in adapted bases), $\text{Sc}\left(C\mathring{\Gamma}\right)$ is the *scalar curvature*, $\mathcal{K}$ is the *Einstein constant* and $\mathcal{T}$ is the intrinsic *stress-energy* d-tensor of matter.

Thus, working with adapted basis of vector fields (4.1), we find the local Einstein geometrical equations for the t-deformed Berwald-Moór metric (3.1). Firstly, by direct computations, we find:

Lemma 5.1. *The Ricci d-tensor of the Cartan canonical connection $C\mathring{\Gamma}$ of the t-deformation of the Berwald-Moór metric (given by (3.1)) has the following effective adapted local Ricci d-tensors:*

$$\begin{gathered} R_{ij} = R^m_{ijm} = \frac{\varkappa^1_{11}\varkappa^1_{11}}{4}S^{(1)(1)}_{(i)(j)}, \quad P^{\ \ (1)}_{i(j)} = P^{(1)}_{(i)j} = P^{m\ (1)}_{ij(m)} = \frac{\varkappa^1_{11}}{2}S^{(1)(1)}_{(i)(j)}, \\ S^{(1)(1)}_{(i)(j)} = S^{m(1)(1)}_{i(j)(m)} = \frac{3\delta_{ij}-1}{9}\cdot\frac{1}{y^i_1 y^j_1} \ \textit{(no sum by } i \textit{ or } j\textit{)}. \end{gathered} \tag{5.3}$$

Remark 5.2. *The vertical Ricci d-tensor $S^{(1)(1)}_{(i)(j)}$ has the following expression:*

$$S^{(1)(1)}_{(i)(j)} = \begin{cases} -\dfrac{1}{9}\dfrac{1}{y^i_1 y^j_1}, & i \neq j \\ \dfrac{2}{9}\dfrac{1}{(y^i_1)^2}, & i = j. \end{cases}$$

Remark 5.3. *Using the last equality of (5.3) and the relation (3.3), we deduce that the following equality is true (sum by r):*

$$S^{m11}_i \stackrel{def}{=} g^{mr}S^{(1)(1)}_{(r)(i)} = G^{-2/3}_{111}\cdot\frac{1-3\delta^m_i}{3}\cdot\frac{y^m_1}{y^i_1} \ \textit{(no sum by } i \textit{ or } m\textit{)}. \tag{5.4}$$

Moreover, by a direct calculation, we obtain the equalities

$$\sum_{m,r=1}^{3} S^{m11}_r C^{r(1)}_{i(m)} = 0, \quad \sum_{m=1}^{3}\frac{\partial S^{m11}_i}{\partial y^m_1} = \frac{2}{3}\cdot\frac{1}{y^i_1}\cdot G^{-2/3}_{111}. \tag{5.5}$$

Lemma 5.4. *The scalar curvature of the Cartan canonical connection* $C\mathring{\Gamma}$ *of the t-deformed Berwald-Moór metric (3.1) is given by*

$$Sc\ \left(C\mathring{\Gamma}\right) = -\frac{4h_{11} + \varkappa^1_{11}\varkappa^1_{11}}{2} \cdot G_{111}^{-2/3}.$$

Proof. The general formula for the scalar curvature of a Cartan connection is (for more details, see [12])

$$\mathrm{Sc}\ \left(C\mathring{\Gamma}\right) = g^{pq}R_{pq} + h_{11}g^{pq}S^{(1)(1)}_{(p)(q)}.$$

□

Describing the global Einstein geometrical equations (5.2) in adapted basis of vector fields (4.1), we find the following important geometrical result (for more details, see [12]):

Proposition 5.5. *The adapted local* ***Einstein geometrical equations****, that govern the non-isotropic gravitational potential (5.1), are given by:*

$$\begin{cases} \xi_{11} \cdot G_{111}^{-2/3} \cdot h_{11} = \mathcal{T}_{11} \\ \dfrac{\varkappa^1_{11}\varkappa^1_{11}}{4\mathcal{K}} S^{(1)(1)}_{(i)(j)} + \xi_{11} \cdot G_{111}^{-2/3} \cdot g_{ij} = \mathcal{T}_{ij} \\ \dfrac{1}{\mathcal{K}} S^{(1)(1)}_{(i)(j)} + \xi_{11} \cdot G_{111}^{-2/3} \cdot h^{11} \cdot g_{ij} = \mathcal{T}^{(1)(1)}_{(i)(j)} \end{cases} \tag{5.6}$$

$$\begin{cases} 0 = \mathcal{T}_{1i}, \quad 0 = \mathcal{T}_{i1}, \quad 0 = \mathcal{T}^{(1)}_{(i)1}, \\ 0 = \mathcal{T}_{1\overset{(1)}{(i)}}, \quad \dfrac{\varkappa^1_{11}}{2\mathcal{K}} S^{(1)(1)}_{(i)(j)} = \mathcal{T}_{i\overset{(1)}{(j)}}, \quad \dfrac{\varkappa^1_{11}}{2\mathcal{K}} S^{(1)(1)}_{(i)(j)} = \mathcal{T}^{(1)}_{(i)j}, \end{cases} \tag{5.7}$$

where

$$\xi_{11} = \frac{4h_{11} + \varkappa^1_{11}\varkappa^1_{11}}{4\mathcal{K}}. \tag{5.8}$$

Remark 5.6. *The Einstein geometrical equations (5.6) and (5.7) impose the stress-energy d-tensor of matter* $\mathcal{T}$ *to be symmetric. In other words, the stress-energy d-tensor of matter* $\mathcal{T}$ *must verify the local symmetry conditions*

$$\mathcal{T}_{AB} = \mathcal{T}_{BA}, \quad \forall\ A, B \in \left\{1,\ i,\ \overset{(1)}{(i)}\right\}.$$

By direct computations, the adapted local Einstein geometrical equations (5.6) and (5.7) imply the following identities of the distinguished stress-energy tensor (summation by r):

$$\mathcal{T}^1_1 \overset{def}{=} h^{11}\mathcal{T}_{11} = \xi_{11} \cdot G_{111}^{-2/3}, \quad \mathcal{T}^m_1 \overset{def}{=} g^{mr}\mathcal{T}_{r1} = 0,$$

$$\mathcal{T}^{(m)}_{(1)1} \overset{def}{=} h_{11}g^{mr}\mathcal{T}^{(1)}_{(r)1} = 0, \quad \mathcal{T}^1_i \overset{def}{=} h^{11}\mathcal{T}_{1i} = 0,$$

$$\mathcal{T}^m_i \overset{def}{=} g^{mr}\mathcal{T}_{ri} = \frac{\varkappa^1_{11}\varkappa^1_{11}}{4\mathcal{K}} S^{m11}_i + \xi_{11} \cdot G_{111}^{-2/3} \cdot \delta^m_i,$$

$$\mathcal{T}^{(m)}_{(1)i} \overset{def}{=} h_{11}g^{mr}\mathcal{T}^{(1)}_{(r)i} = \frac{h_{11}\varkappa^1_{11}}{2\mathcal{K}} S^{m11}_i, \quad \mathcal{T}^{1(1)}_{\ \ (i)} \overset{def}{=} h^{11}\mathcal{T}_{1\overset{(1)}{(i)}} = 0,$$

$$\mathcal{T}^{m(1)}_{\ \ (i)} \overset{def}{=} g^{mr}\mathcal{T}_{r\overset{(1)}{(i)}} = \frac{\varkappa^1_{11}}{2\mathcal{K}} S^{m11}_i,$$

$\mathcal{T}^{(m)(1)}_{(1)(i)} \stackrel{def}{=} h_{11}g^{mr}\mathcal{T}^{(1)(1)}_{(r)(i)} = \dfrac{h_{11}}{\mathcal{K}}S^{m11}_i + \xi_{11}\cdot G_{111}^{-2/3}\cdot\delta^m_i$, where the distinguished tensor S^{m11}_i is given by (5.4) and ξ_{11} is given by (5.8).

Proposition 5.7. *The stress-energy d-tensor of matter $\mathcal{T}$ must verify the following* ***conservation geometrical laws*** *(summation by m):*

$$\begin{cases} \mathcal{T}^1_{1/1} + \mathcal{T}^m_{1|m} + \mathcal{T}^{(m)}_{(1)1}|^{(1)}_{(m)} = \dfrac{(h^{11})^2}{16\mathcal{K}}\dfrac{dh_{11}}{dt}\left[2\dfrac{d^2h_{11}}{dt^2} - \dfrac{3}{h_{11}}\left(\dfrac{dh_{11}}{dt}\right)^2\right]\cdot G_{111}^{-2/3} \\ \mathcal{T}^1_{i/1} + \mathcal{T}^m_{i|m} + \mathcal{T}^{(m)}_{(1)i}|^{(1)}_{(m)} = 0 \\ \mathcal{T}^{1(1)}_{\ \ (i)/1} + \mathcal{T}^{m(1)}_{\ \ (i)|m} + \mathcal{T}^{(m)(1)}_{(1)(i)}|^{(1)}_{(m)} = 0, \end{cases}$$

where (summation by m and r)

$$\mathcal{T}^1_{1/1} \stackrel{def}{=} \frac{\delta\mathcal{T}^1_1}{\delta t} + \mathcal{T}^1_1\varkappa^1_{11} - \mathcal{T}^1_1\varkappa^1_{11} = \frac{\delta\mathcal{T}^1_1}{\delta t},$$

$$\mathcal{T}^m_{1|m} \stackrel{def}{=} \frac{\delta\mathcal{T}^m_1}{\delta x^m} + \mathcal{T}^r_1 L^m_{rm} = \frac{\delta\mathcal{T}^m_1}{\delta x^m},$$

$$\mathcal{T}^{(m)}_{(1)1}|^{(1)}_{(m)} \stackrel{def}{=} \frac{\partial\mathcal{T}^{(m)}_{(1)1}}{\partial y^m_1} + \mathcal{T}^{(r)}_{(1)1}C^{m(1)}_{r(m)} = \frac{\partial\mathcal{T}^{(m)}_{(1)1}}{\partial y^m_1},$$

$$\mathcal{T}^1_{i/1} \stackrel{def}{=} \frac{\delta\mathcal{T}^1_i}{\delta t} + \mathcal{T}^1_i\varkappa^1_{11} - \mathcal{T}^1_r G^r_{i1} = \frac{\delta\mathcal{T}^1_i}{\delta t} + \mathcal{T}^1_i\varkappa^1_{11},$$

$$\mathcal{T}^m_{i|m} \stackrel{def}{=} \frac{\delta\mathcal{T}^m_i}{\delta x^m} + \mathcal{T}^r_i L^m_{rm} - \mathcal{T}^m_r L^r_{im} = \frac{\varkappa^1_{11}}{2}\frac{\partial\mathcal{T}^m_i}{\partial y^m_1},$$

$$\mathcal{T}^{(m)}_{(1)i}|^{(1)}_{(m)} \stackrel{def}{=} \frac{\partial\mathcal{T}^{(m)}_{(1)i}}{\partial y^m_1} + \mathcal{T}^{(r)}_{(1)i}C^{m(1)}_{r(m)} - \mathcal{T}^{(m)}_{(1)r}C^{r(1)}_{i(m)} = \frac{\partial\mathcal{T}^{(m)}_{(1)i}}{\partial y^m_1},$$

$$\mathcal{T}^{1(1)}_{\ \ (i)/1} \stackrel{def}{=} \frac{\delta\mathcal{T}^{1(1)}_{\ \ (i)}}{\delta t} + 2\mathcal{T}^{1(1)}_{\ \ (i)}\varkappa^1_{11},$$

$$\mathcal{T}^{m(1)}_{\ \ (i)|m} \stackrel{def}{=} \frac{\delta\mathcal{T}^{m(1)}_{\ \ (i)}}{\delta x^m} + \mathcal{T}^{r(1)}_{\ \ (i)}L^m_{rm} - \mathcal{T}^{m(1)}_{\ \ (r)}L^r_{im} = \frac{\varkappa^1_{11}}{2}\frac{\partial\mathcal{T}^{m(1)}_{\ \ (i)}}{\partial y^m_1},$$

$$\mathcal{T}^{(m)(1)}_{(1)(i)}|^{(1)}_{(m)} \stackrel{def}{=} \frac{\partial\mathcal{T}^{(m)(1)}_{(1)(i)}}{\partial y^m_1} + \mathcal{T}^{(r)(1)}_{(1)(i)}C^{m(1)}_{r(m)} - \mathcal{T}^{(m)(1)}_{(1)(r)}C^{r(1)}_{i(m)} = \frac{\partial\mathcal{T}^{(m)(1)}_{(1)(i)}}{\partial y^m_1}.$$

Proof. The above conservation geometrical laws are provided by direct computations, using the relations (4.3) and (5.5). □

5.2 Electromagnetic-like geometrical model

In the paper [12], using only a given Lagrangian function $L(t,x,y)$ on the 1-jet space $J^1(\mathbb{R},M^n)$, an electromagnetic-like geometrical model was also created. In the background of our electromagnetic-like geometrical formalism from [12], we work with an *electromagnetic distinguished* 2*-form* (the Latin letters run from 1 to n)

$$\mathbb{F}=F_{(i)j}^{(1)}\delta y_1^i\wedge dx^j,$$

where

$$F_{(i)j}^{(1)}=\frac{h^{11}}{2}\left[g_{jm}N_{(1)i}^{(m)}-g_{im}N_{(1)j}^{(m)}+\left(g_{ir}L_{jm}^r-g_{jr}L_{im}^r\right)y_1^m\right].$$

The electromagnetic components $F_{(i)j}^{(1)}$ are characterized by the following *Maxwell geometrical equations* [12]:

$$\begin{aligned}F_{(i)j/1}^{(1)} &= \frac{1}{2}\mathcal{A}_{\{i,j\}}\left\{\overline{D}_{(i)1|j}^{(1)}-D_{(i)m}^{(1)}G_{j1}^m+d_{(i)(m)}^{(1)(1)}R_{(1)1j}^{(m)}-\right.\\ &\left.-\left[C_{j(m)}^{p(1)}R_{(1)1i}^{(m)}-G_{i1|j}^p\right]h^{11}g_{pq}y_1^q\right\},\end{aligned}$$

$$\sum_{\{i,j,k\}}F_{(i)j|k}^{(1)}=-\frac{1}{4}\sum_{\{i,j,k\}}\frac{\partial^3L}{\partial y_1^i\partial y_1^p\partial y_1^m}\left[\frac{\delta N_{(1)j}^{(m)}}{\delta x^k}-\frac{\delta N_{(1)k}^{(m)}}{\delta x^j}\right]y_1^p,$$

$$\sum_{\{i,j,k\}}F_{(i)j}^{(1)}|_{(k)}^{(1)}=0,$$

where $\mathcal{A}_{\{i,j\}}$ means an alternate sum, $\sum_{\{i,j,k\}}$ means a cyclic sum and we have

$$\overline{D}_{(i)1}^{(1)}=\frac{h^{11}}{2}\frac{\delta g_{im}}{\delta t}y_1^m,\quad D_{(i)j}^{(1)}=h^{11}g_{ip}\left[-N_{(1)j}^{(p)}+L_{jm}^py_1^m\right],$$

$$d_{(i)(j)}^{(1)(1)}=h^{11}\left[g_{ij}+g_{ip}C_{m(j)}^{p(1)}y_1^m\right],$$

$$\overline{D}_{(i)1|j}^{(1)}=\frac{\delta\overline{D}_{(i)1}^{(1)}}{\delta x^j}-\overline{D}_{(m)1}^{(1)}L_{ij}^m,\quad G_{i1|j}^k=\frac{\delta G_{i1}^k}{\delta x^j}+G_{i1}^mL_{mj}^k-G_{m1}^kL_{ij}^m,$$

$$F_{(i)j/1}^{(1)}=\frac{\delta F_{(i)j}^{(1)}}{\delta t}+F_{(i)j}^{(1)}\varkappa_{11}^1-F_{(m)j}^{(1)}G_{i1}^m-F_{(i)m}^{(1)}G_{j1}^m,$$

$$F_{(i)j|k}^{(1)}=\frac{\delta F_{(i)j}^{(1)}}{\delta x^k}-F_{(m)j}^{(1)}L_{ik}^m-F_{(i)m}^{(1)}L_{jk}^m,$$

$$F_{(i)j}^{(1)}|_{(k)}^{(1)}=\frac{\partial F_{(i)j}^{(1)}}{\partial y_1^k}-F_{(m)j}^{(1)}C_{i(k)}^{m(1)}-F_{(i)m}^{(1)}C_{j(k)}^{m(1)}.$$

Example 5.8. *The Lagrangian function that governs the movement law of a particle of mass $m\neq 0$ and electric charge e, which is displaced concomitantly into an environment endowed both with a gravitational field and an electromagnetic one, is given by*

$$L(t,x^k,y_1^k)=mch^{11}(t)\,\varphi_{ij}(x^k)\,y_1^iy_1^j+\frac{2e}{m}A_{(i)}^{(1)}(t,x^k)\,y_1^i+\mathcal{F}(t,x^k),\tag{5.9}$$

where the semi-Riemannian metric $\varphi_{ij}(x)$ represents the ***gravitational potential*** *of the space of events M, $A_{(i)}^{(1)}(t,x)$ are the components of a d-tensor on the 1-jet space $J^1(\mathbb{R},M)$ representing the* ***electromagnetic potential*** *and $\mathcal{F}(t,x)$ is a smooth* ***potential function*** *on the* *product manifold* $\mathbb{R}\times M$. *It is important to note that the jet Lagrangian function (5.9) is a natural extension of the Lagrangian (defined on the tangent bundle) used in electrodynamics by Miron and Anastasiei [8]. In our jet Lagrangian formalism applied to (5.9), the* ***electromagnetic components*** *are given by (see [12])*

$$F_{(i)j}^{(1)} = -\frac{e}{2m}\left(\frac{\partial A_{(i)}^{(1)}}{\partial x^j} - \frac{\partial A_{(j)}^{(1)}}{\partial x^i}\right),$$

and the second set of ***Maxwell geometrical equations*** *reduce to the classical ones [12]:*

$$\sum_{\{i,j,k\}} F_{(i)j|k}^{(1)} = 0,$$

where

$$F_{(i)j|k}^{(1)} = \frac{\partial F_{(i)j}^{(1)}}{\partial x^k} - F_{(m)j}^{(1)}\gamma_{ik}^m - F_{(i)m}^{(1)}\gamma_{jk}^m.$$

This fact suggests, in our opinion, some kind of naturalness attached to our electromagnetic-like geometrical theory.

On our particular 1-jet space $J^1(\mathbb{R}, M^3)$, the t-deformed Berwald-Moór metric (3.1) and the nonlinear connection (3.7) produce the electromagnetic 2-form

$$\mathbb{F} := \mathring{\mathbb{F}} = 0.$$

In conclusion, our t-deformed Berwald-Moór electromagnetic-like geometrical model on the 1-jet three-dimensional time is trivial. In other words, in our jet geometrical approach, the t-deformed Berwald-Moór electromagnetism (produced by (3.1) and (3.7)) leads us to null electromagnetic geometrical components and to tautological Maxwell-like equations. In our opinion, this fact suggests that the t-deformed Berwald-Moór geometrical structure of the 1-jet three-dimensional time contains rather gravitational connotations than electromagnetic ones. In such a perspective, it seems that we need to consider a similar geometrical study for x-dependent conformal deformations of the Berwald-Moór structure, agreeing thus with the recent geometric-physical ideas proposed by Garas'ko in [4].

Acknowledgements.

The authors of this paper express their gratefulness to the referee of Hypercomplex Numbers in Geometry and Physics for his valuable comments and many useful suggestions.

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ЛОКАЛЬНАЯ РИМАНОВО-ФИНСЛЕРОВА ГЕОМЕТРИЯ СТРУЙ ДЛЯ ТРЕХМЕРНОГО ВРЕМЕНИ

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Целью настоящей работы является развитие 1-стуйного пространства финслероподобной геометрии (в смысле отмеченной (d-) связности, d-кручения и d-кривизны) для реономной метрики Бервальда-Моора третьего порядка (т.е. времени-зависимых конформных деформаций обычных струй Бервальда-Моора или метрики третьего порядка). Также приведены некоторые естественные геометрические теории поля (гравитация и элетромагнетизм) следующие из этой реономной метрики Бервальда-Моора.

Ключевые слова: реономная метрика Бервальда-Моора третьего порядка, каноническая нелинейная связность, каноническая связность Картана, d-кручение и d-кривизна, геометрические уравнения Эйнштейна.
MSC(2000): 53C60, 53C80, 83C22.