DIFFERENTIAL FORMS: FROM CLIFFORD, THROUGH CARTAN TO KÄHLER

Jose G. Vargas

University of South Carolina,Columbia, SC, USA
josegvargas@earthlink.net

Limitations of the vector, tensor and Dirac calculi are illustrated to motivate the Kaehler calculus of integrands, which replaces all three of them and which we introduce in three steps.

In a first step, we present the basics of the underlying Clifford algebra for that calculus, algebra valid for Euclidean and pseudo-Euclidean vector spaces of arbitrary dimension. The usual vector algebra is shown to be a corrupted form of Clifford algebra, corruption specific to dimension three and non-existing for other dimensions. The Clifford product is constituted by the sum of the exterior and interior products if at least one of the factors is a vector. Grossly speaking, these products play the role of the vector and scalar products of three dimensions, while generalizing them. It thus contains exterior algebra.

As an intermediate step towards the Kaehler calculus, we briefly give the fundamentals of Cartan's exterior calculus of scalar-valued differential forms, here viewed as ordinary scalar-valued integrands in multiple integrals. We also make a brief incursion into the exterior calculus of vector-valued differential forms, which is the moving frame version of differential geometry.

We show the basics of the Kaehler calculus of differential forms. It is to the exterior calculus what Clifford algebra is to exterior algebra. Because of time and complexity constraints, we limit ourselves to scalar-valued differential forms, which is sufficient for relativistic quantum mechanics with electromagnetic coupling. In using this calculus, the problem with negative energy-solutions does not arise.

Key Words: differential forms, hypercomplex numbers, Clifford algebra, exterior calculus, moving frames, Kähler calculus.

1 Introduction

The calculus of differential forms is a formidable mathematical tool whose reach has been underestimated even by its advocates. Usually associated with E. Cartan since he designed it in 1899 [1], its most advanced expression took place with the work of Kähler in the early sixties [2],[3],[4] (all of them in German). At that point differential forms, which had been the most sophisticated language for differential geometry and general relativity, became also the most sophisticated language for quantum physics (See Ref. [5] for justification of this claim).

Our report should begin with the seminal work of Grassmann in 1844 [6]. Considerations of space dictate, however, that we start with the work of Clifford in the early 1870's [7], given that the algebra that bears his name underlies the Kähler calculus. Clifford algebra is to the Kähler calculus what exterior algebra is to the Cartan calculus. All the basic ideas in Clifford were, however, already present in Grassmann, arguably the greatest mathematical genius of the nineteenth century (See [8] and [9] for how incredibly ahead of his epoch Grassmann was).

Structurally, Clifford algebra should carry the name of Euclid, if not of Grassmann. This may sound disconcerting at first, since the Clifford algebra Cl^3 for three-dimensional Euclidean space, E^3, is not the one taught in school when studying Euclidean geometry. The latter, however, can be dissected and restructured, resulting in Cl^3, as will be shown in section 3. No algebras like the school algebra exist in Euclidean spaces of arbitrary dimension. Clifford algebra on the other hand is valid in spaces of arbitrary dimension.

Different authors have shown that, to a large extent, one can dispense with the vector and tensor calculi, specially if one complements the exterior derivative with the coderivative. In a book in progress [10], this author shows that the more comprehensive use of differential forms that the Kähler calculus makes allows one to totally dispense with those two calculi. In this paper, we shall make the point that it also allow us to dispense with the Dirac calculus with gamma matrices. But it is the replacement of the calculus of complex variable that is specifically suitable to illustrate the nature of the Kähler calculus as hyper complex.

After this author delivered this paper at the official presentation of the Research Institute on Hyper Complex Systems in Geometry and Physics (May 4-5, 2009 in Moscow at Bauman University and Fryazino, Russia), he showed that the calculus of complex variable may be equally unnecessary [11]. To be precise, the use of the Kähler calculus of differential forms in the real plane permits an enormous simplification of the calculus of complex variable by replacing the concept of analytic complex variable functions with the concept of strict harmonic even differential forms. Further yet, the same calculus permits us to ignore Cauchy's theory of integration and replace it with a revision of Stoke's theorem. The almost two centuries old Cauchy's theory was retrospectively an extension of the at the time unborn Stoke's theorem (1854). Cauchy thus produced a solution as genial as unnatural to a problem whose time had not yet come, much less had the modern language of differentiation and integration been born. Notice that said theorem refers to surfaces without poles. But this is an artificial limitation because the integrand $d\phi$ (where ϕ is the polar angle) has no poles in the (ρ, ϕ) plane, but $(xdy - ydx)/(x^2 + y^2)$, which is the pull-back of $d\phi$ by a coordinate transformation, does. This observation makes one discover the role of $x + ydxdy$ as the complex variable z in the Kähler calculus, where the theorem of residues becomes a simple corollary to the combination of the Fourier series expansion and the standard Stokes theorem. This theorem is thus generalized without resort to Cauchy's theory. The algebra that allows for the replacement with great advantage of the calculus of complex variable is a hyper complex algebra with four independent units whose squares are (1,1,1,-1).

The view of differential forms as integrands in multiple integrals is essential for understanding this paper. In other words, differential forms are functions of r-surfaces; a 1-surface is a curve, a 2-surface is a standard surface, a 3-surface is the type of figure to which we ascribe a volume, etc. A most authoritative presentation of the exterior calculus as pertaining to integrands is Rudin's book "Principles of Mathematical Analysis" [12]. One cannot fully understand Cartan, and much less Kähler, unless one has the same view of differential forms as Rudin. One often finds in books on the exterior calculus the definition of differential r-forms as antisymmetric r-linear functions of vectors. At the same time, some of those very same books state in their introductions that differential forms are functions of r-surfaces. That leads to confusion; different types of functions should have different names.

To facilitate the immersion into the main core of the paper, we present in section 2 a perspective of the different calculi that we have mentioned above. In section 3 and for the same purpose, we exhibit familiar examples of differential forms of different valuednesses. In sections 4 an 5, we deal with the highlights of the work of Clifford and Cartan that are relevant for introducing the Kähler calculus in section 6. This is the one which receives greater attention in this article, since it is the least known. In any case, it is not our intention to do a comprehensive presentation either of Clifford algebra or of the calculi by Cartan and by Kähler.

2 Perspective on alternative calculi

2.1 The horrible vector calculus

In a paper on the legacy and misfortune of genial algebraist Grassmann (whose work was not recognized even by Hilbert and Weyl and who up to this day appears to be largely misunderstood [9]), Dieudonne uses the term "horrible vector calculus" [8]. He does not explain why he does so. This author has found the following examples to justify his characterization:

1. There is no vector product for arbitrary dimension (Lounesto reports that, dimension seven also is an exception [13], but notice that there is then a whole five-dimensional subspace perpendicular to the plane of two vectors in a vector product). Without a vector product, there is not a vector calculus.

2. To be specific, there is no curl and, therefore, no curl theorems without a vector product, unless, of course, one does equivalent theory in a totally different way.

3. A missing curl-related theorem would deal with the integrability of

$$\int_A^B fdx + gdy + hdz + ... + jdt,$$

i.e, with whether that integral between two points A and B depends on path or not.

4. The vector calculus (we are thus returning to three dimensions) is contrived, abstruse. This is all too obvious in retrospect if, in the process of computing with curvilinear coordinates, we reach an integral even as simple as

$$\int_A^B \rho d\phi$$

and we ask ourselves whether this integral between points A and B depends on path. We again have to resort to the curl but, who can ever remember curls in curvilinear coordinate systems? Or we might first change the line integrand $\rho d\phi$ to Cartesian coordinates, and then extract the components $(f, g, 0)$ of a vector field whose curl we would then compute in those coordinates. Any of these two processes is unnecessarily laborious. The vector calculus misleads one into the line of argument just provided when, in fact, the nature of ρ and ϕ as curvilinear coordinates is totally irrelevant for this problem. With the calculus of differential forms, it becomes immediately obvious that the integral under consideration is path dependent.

5. The vector calculus is misleading also in a different way. If we are given the gradient of a function, the difference in the value at two points of this function is given by

$$F(B) - F(A) = \int_A^B \nabla F.d\mathbf{r} = \int_A^B f dx + g dy + h dz,$$

where (f, g, h) are the partial derivatives of F. Notice that the dot product is present twice in $\nabla F.d\mathbf{r}$. Apart from the explicit appearance, an implicit one resides in the dependence of the gradient on the metric, which is the product $d\mathbf{r}.d\mathbf{r}$. These two appearances cancel each other out. $\int_A^B \nabla F.d\mathbf{r}$ thus uses concepts which requires more structure than is necessary to obtain $F(B) - F(A)$. The role of the gradient is played by $f dx + g dy + h dz$ (for the pertinent functions, f, g and h). This expression pertains even to manifolds where a metric is not even defined.

6. The vector calculus is unnatural. Consider a change of variables of multiple integrals even as simple as

$$\int_R dx\, dy.$$

A coordinate substitution, say to polar coordinates, yields

$$dx = d\rho \cos\phi - \rho \sin\phi d\phi, \qquad dy = d\rho \sin\phi + \rho \cos\phi d\phi.$$

Substitution of these equations in $dxdy$ yields an expression which does not even make sense as an integrand, given the presence of symmetric quadratic terms in the differential of the coordinates. If, on the other hand, one writes the integrand with the appropriate product, $dx \wedge dy$, the same substitution will produce the right integrand in polar coordinates without resort to the Jacobian, which emerges automatically. And this mechanism works for any r-integrand in n dimensions, and any differentiable change of variables.

Any book on the vector calculus gives the process by which one obtains the correct expression for the integrand in terms of new coordinates. But it does not answer the question of why the aforementioned substitution fails to work for $dxdy$, a fact that goes against the experience by students of correctly performing such substitutions when dealing with line integrals. The failure now is due to the fact that the (tensor) product $dxdy$ is not the right one. What is needed is the so called exterior product, $dx \wedge dy$, to be considered later.

The inadequacy of vector algebra is addressed by Clifford's algebra, though, again, Grassmann did the main work leading to it.

2.2 The tensor calculus

The tensor calculus was developed virtually in parallel with the vector calculus, responding to the need to work with dimensions greater than three and in curved spaces. It evolved as a collection of concepts, rules, theorems, etc., which Ricci and Levi-Civita put together [14]. The formal development of the underlying tensor algebra seems to have occurred later. Although the tensor calculus is not limited to three dimensions, it nevertheless lacks the sophistication required as an instrument for research in differential geometry. More importantly, it is useless for relativistic quantum mechanics, which is the reason why the Dirac calculus was developed.

The aforementioned lack of sophistication manifests itself, for example, in that the different nature of the indices of the curvature is obscured. It is not sufficient to know that a set of quantities transforms tensorially, since transformation properties do not speak of the nature of the objects whose components are those quantities. For instance the same transformation rule applies to the components of a line integrand, as to the components of a field of a linear functions of vector or, if the space or manifold is endowed with a metric, as to the components of a vector field referred to what is known as a field of reciprocal vector bases. Regarding the components of the curvature, the last two of its subscripts pertain to its being a differential 2-form, the other subscript belongs to its valuedness as a linear function of vectors, and its superscript concerns its valuedness also as a tangent vector field.

Even more important is the fact that connections —which are pariah in the tensor calculus since they do not transform tensorially— are full fledged Lie algebra-valued differential 1-forms. Tensors still play significant but unnecessary roles even in otherwise sophisticated books on the geometry of physics where differential forms are used but with heavy involvement of tensorial concepts. In those books, Yang-Mills connections are introduced as being Lie algebra valued, but this is not done, unfortunately, with the classical connections. Thus the wrong impression is given that Lie algebra valuedness pertains exclusively to Yang Mills theory. This lack of sophistication, with potentially damaging implications for theoretical physics, is easily avoidable in a calculus of differential forms if and when it virtually excludes any role for tensors.

2.3 The Dirac calculus

The Dirac calculus is an ad hoc response to the needs of relativistic quantum mechanics at a time when the calculus was not sufficiently developed. Dirac's use of the so called gamma matrices is, retrospectively, totally unnecessary; their Clifford algebraic structure suffices, irrespective of their nature as matrices. The Kähler calculus is the calculus of differential forms consistent with an underlying Clifford algebra. Since the exterior structure is contained in the Clifford structure, the exterior calculus is contained in the Kähler calculus, which thus serves both, general relativity and quantum mechanics.

Another great advantage of the Kähler calculus over Dirac's is that the Kähler field is more like a classical field than like a spinor field. This is no way means that spinors are not relevant. They remain very much so. But they are a derived concept that emerges from the Kähler field as a result of the sophisticated nature of the proper value solutions of differential systems expressed in terms of differential forms.

Finally, the negative energies associated with antiparticles are a spurious effect of the Dirac theory, as antiparticles emerge nicely with the same energy as the respective particles in the Kähler calculus. Thus, nothing as contrived as hole theory is needed.

3 Types of differential forms

For a practical acquaintance with various types of differential forms, we proceed to consider some familiar expressions. Let f, g and h be functions of the (x, y, z) coordinates. The expression

$$f dx + g dy + h dz$$

is a scalar-valued differential 1-form. The "one" in 1-form has to do with the fact that this is a linear function of the differential of the coordinates. For comparison,

$$x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

is said to be a differential 0-form, since there are no differentials in it. The presence of vectors in this expressions makes it a vector-value 0-form. Clearly, then,

$$dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k}$$

is a vector-valued differential 1-form, rather than a vector, as it is said to be in the vector calculus. Let us rewrite the last expressions as

$$\sum_{p,q} \delta_p^q \, dx^p \, \mathbf{a}_q$$

where $\mathbf{a}_q = (\mathbf{i}, \mathbf{j}, \mathbf{k})$, and where δ_p^q is the Kronecker delta.

The scalar-valued functions of several variables are scalar-valued differential 0-forms. Their grade as differential forms is zero.

The integrand $dx\,dy$, which should actually be written as $dx \wedge dy$, is a function of surfaces, i.e. a scalar-valued differential 2-form (grade 2). The "2" in 2-form obviously has to do with the fact that it is a quadratic expression in the differential of the coordinates. Magnetic fields are differential 2-forms:

$$B_x dy \wedge dz + B_y dz \wedge dx + B_z dx \wedge dy.$$

A volume integrand

$$dx\,dy\,dz \to dx \wedge dy \wedge dz \leftrightarrow r \sin^2\theta \, dr \wedge d\theta \wedge d\phi$$

is a differential 3-form. Similarly,

$$f\,\mathbf{j} \wedge \mathbf{k} + g\,\mathbf{k} \wedge \mathbf{i} + \, h\,\mathbf{i} \wedge \mathbf{j}$$

is a bivector-valued differential 0-form. Both $dx \wedge dy$ and $\mathbf{j} \wedge \mathbf{k}$ are objects of grade two, each in its respective algebra. Although differential 0-forms may appear to have nothing to do with integration and/or differentiation, they do. We shall understand this when we later deal with the general Stokes theorem.

4 Clifford

4.1 Clifford view of standard vector algebra

The usual vector algebra in E^3 is a corruption of Clifford algebra, as we now explain. The vector product is a combination of two operations, respectively called exterior product and Hodge duality. The latter consists in assigning to each object of grade 2 an object of grade $n-2$ in the same algebra, where n is the dimensionality of the space. The exterior product of two vectors is of grade two (a bivector). In E^3, the duality operation on that exterior product yields an object of grade one, which is the vector product. In dimension other than three, the dual of the exterior product is not of grade one, i.e. it is not a vector. As we already said, a meaningful concept of vector product does not exist in arbitrary dimension. On the other hand, exterior products exist in any number of dimensions. In obtaining the vector product in three dimensions, we take the step of Hodge duality after the exterior product. Again, Hodge duality does not return a vector in other dimensions. That would not be too bad by itself, were it not for the fact that it is not explicitly mentioned in the vector calculus and does not, therefore, prompt one to think about how one could address the whole issue of the vector product and its replacement in an arbitrary number of dimensions.

The exterior product of two vectors can be put together with their dot product to yield the so called Clifford product, its antisymmetric and symmetric parts being the exterior and dot (also called interior) products respectively. Notice that this gives the sum of a bivector and a scalar, for any n. We shall later find that this pattern is at the root of the (replacements for) curl and divergence coming together.

4.2 Basic products in Clifford algebra

Let us now proceed in the opposite direction, from the Clifford product to the exterior and dot products. Let $\mathbf{a}$, $\mathbf{b}$...be vectors of a space (say over the reals) endowed with a dot product. One may construct an algebraic structure known as Clifford algebra. The binary operations with vectors are condensed in the formulas

$$\mathbf{ab} \equiv \mathbf{a} \vee \mathbf{b} = \mathbf{a} \wedge \mathbf{b} + \mathbf{a} \cdot \mathbf{b},$$

where $\vee$ (or just juxtaposition of vectors) means Clifford product. The exterior product is defined as the antisymmetric part of $\mathbf{ab}$

$$\mathbf{a} \wedge \mathbf{b} \equiv \frac{1}{2}(\mathbf{ab} - \mathbf{ba}),$$

and it is what, in E^3, we identified as the vector product minus the Hodge duality. The interior product is then defined as the symmetric part

$$\mathbf{a} \cdot \mathbf{b} \equiv \frac{1}{2}(\mathbf{ab} + \mathbf{ba}),$$

and is identified with the standard scalar product. These equations imply

$$\mathbf{a} \vee \mathbf{b} + \mathbf{b} \vee \mathbf{a} = 2\mathbf{a} \cdot \mathbf{b}.$$

If $\mathbf{a}$ and $\mathbf{b}$ are elements of a basis, this equation becomes

$$\mathbf{a}_i \vee \mathbf{a}_j + \mathbf{a}_\mathbf{i} \vee \mathbf{a}_\mathbf{j} = 2g_{ij},$$

where g_{ij} is defined as $\mathbf{a}_i \cdot \mathbf{a}_j$.

The exterior and interior parts are of respective grades two and zero. One can multiply any number of vectors and get sums of quantities of different grades. In contrast, the grade of the exterior product is the sum of the grades of the factors in the product, if not zero. Thus, for example, the exterior product of a vector and a bivector is, when not zero, a trivector, which is of grade 3.

Readers overwhelmed by this structural richness can help themselves by thinking of $\mathbf{a}$ and $\mathbf{b}$ as two gamma matrices, their Clifford product being matrix multiplication, but without actually carrying out explicit multiplication of matrices, but leaving them indicated.

Let us show how the algebra of three-dimensional Euclidean space differs from ordinary vector algebra. In this section, we focus on the exterior product, as the dot or interior product is sufficiently known. Because of skew symmetry (also referred to as antisymmetry), we have

$$\mathbf{i} \wedge \mathbf{i} = \mathbf{j} \wedge \mathbf{j} = \mathbf{0}, \qquad \mathbf{i} \wedge \mathbf{j} = -\,\mathbf{j} \wedge \mathbf{i}.$$

We do not reduce or relate $\mathbf{i} \wedge \mathbf{j}$ to a vector; it is just a unit of grade two (a vector is of grade one). That property, together with the distributive property required of the exterior product, allows us to write

$$\begin{aligned} \mathbf{a} \wedge \mathbf{b} &= (a^1\mathbf{i} + a^2\mathbf{j} + a^3\mathbf{k}) \wedge (b^1\mathbf{i} + b^2\mathbf{j} + b^3\mathbf{k}) = \\ &= (a^2b^3 - a^3b^2 ...)\mathbf{j} \wedge \mathbf{k} + (a^3b^1 - a^1b^3 ...)\mathbf{k} \wedge \mathbf{i} + (a^1b^2 - a^2b^1)\mathbf{i} \wedge \mathbf{j}. \end{aligned}$$

The exterior product of a number r of independent vectors is called an r-vector. The exterior and dot products of an r-vector by a vector yield an $(r+1)-$vector and $(r-1)-$vector,respectively. In dealing with matrices, their explicit multiplication hides all this structure. Hence, in learning Clifford algebra, one should help oneself with matrices only temporarily, and to the least possible extent. Notice that the components of $\mathbf{a} \wedge \mathbf{b}$ relative to $\mathbf{j} \wedge \mathbf{k}$, $\mathbf{k} \wedge \mathbf{i}$ and $\mathbf{i} \wedge \mathbf{j}$ are the same as those of the vector product with respect to $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$.

The exterior product satisfies the associative property also, as in

$$\mathbf{i} \wedge (\mathbf{j} \wedge \mathbf{k}) = (\mathbf{i} \wedge \mathbf{j}) \wedge \mathbf{k} = \mathbf{i} \wedge \mathbf{j} \wedge \mathbf{k}.$$

It should be obvious that repeated factors in the exterior product cause it to cancel, as in

$$\mathbf{k} \wedge (\mathbf{i} \wedge \mathbf{j} \wedge \mathbf{k}) = \mathbf{0},$$

because we can put the two $\mathbf{k}$'s together. We similarly have

$$\mathbf{a} \wedge (\mathbf{i} \wedge \mathbf{j} \wedge \mathbf{k}) = \mathbf{0},$$

since $\mathbf{a}$ is a linear combination of $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$.

Let α and β be multivectors of respective grades r and s. Their exterior product will be of dimension $r + s$, if not zero. Thus, for example

$$(\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) \wedge (\mathbf{b} \wedge \mathbf{d}) = \mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} \wedge \mathbf{b} \wedge \mathbf{d} = \mathbf{0},$$

since one factor is repeated.

4.3 Mixing products in the Clifford algebra

We give a sample of products in the Clifford algebra. Let λ represent scalars. We have

$$\mathbf{a}\cdot\mathbf{b}=\lambda, \qquad \mathbf{a}\cdot\lambda=0,$$

$$\mathbf{a}\cdot(\mathbf{b}\wedge\mathbf{c})=-(\mathbf{b}\wedge\mathbf{c})\cdot\mathbf{a},$$

and (notice the alternation of signs)

$$\mathbf{a}\cdot(\mathbf{b}\wedge\mathbf{c}\wedge\mathbf{d}\wedge...)=(\mathbf{a}\cdot\mathbf{b})(\mathbf{c}\wedge\mathbf{d}\wedge...)-(\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\wedge\mathbf{d}\wedge...)+...-...,$$

$$\mathbf{a}\cdot(\mathbf{b}\vee\mathbf{c}\vee\mathbf{d}\vee...)=(\mathbf{a}\cdot\mathbf{b})(\mathbf{c}\vee\mathbf{d}\vee...)-(\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\vee\mathbf{d}\vee...)+...-...$$

The associativity property

$$\mathbf{a}\vee\mathbf{b}\vee\mathbf{c}\vee\mathbf{d}=(\mathbf{a}\vee\mathbf{b})\vee\mathbf{c}\vee\mathbf{d}=\mathbf{a}\vee(\mathbf{b}\vee\mathbf{c})\vee\mathbf{d}=...$$

extends to multivectors,

$$\mathbf{A}\vee\mathbf{B}\vee\mathbf{C}=(\mathbf{A}\vee\mathbf{B})\vee\mathbf{C}=\mathbf{A}\vee(\mathbf{B}\vee\mathbf{C}),$$

where we have used capitals to denote elements of arbitrary and even mixed grade. For a Clifford product of a vector and any other element of the algebra, we have

$$\mathbf{a}\vee\mathbf{A}=\mathbf{a}\wedge\mathbf{A}+\mathbf{a}\cdot\mathbf{A},$$

but not if **a** also were of grade greater than one.

The Clifford product built upon a vector space of dimension n is as well a vector space, specifically of dimension 2^n. There subspaces of scalars, vectors, bivectors, ...r-vectors,...n-vectors constitute respective subspaces. The dimension obviously is the sum of their dimensions. For $n=3:$

$$\tbinom{3}{0}+\tbinom{3}{1}+\tbinom{3}{2}+\tbinom{3}{3}=2^3=8,$$

and, for arbitrary dimension:

$$\tbinom{n}{0}+\tbinom{n}{1}+\tbinom{n}{2}+...\tbinom{n}{n-1}+\tbinom{n}{n}=2^n$$

If the algebra is over the complex numbers, it will be like a real space of dimension 2^{n+1}.

4.4 Computing in automatic pilot

The Clifford-algebraic structure does a "lot of thinking" for us, once we know the rules to multiply in the algebra. An example is the decomposition of a vector **a** into components $\mathbf{a}_\perp$ and $\mathbf{a}_{\shortparallel}$ respectively perpendicular and parallel to another vector **b**.

Multiply **a** and **b** with the Clifford product (the other products loose information) and recover **a** by multiplying **ab** by $\mathbf{b}^{-1}$, which equals $\mathbf{b}/b^2$. Let **n** be the unit vector in the direction of **n**. Each Clifford product gives two terms in an obvious manner, so that we get:

$$\begin{aligned}(\mathbf{ab})\mathbf{b}^{-1} &= (\mathbf{ab})\frac{\mathbf{b}}{b^2}=(\mathbf{an})\mathbf{n}=\\ &= (\mathbf{a}\wedge\mathbf{n})\wedge\mathbf{n}+(\mathbf{a}\wedge\mathbf{n})\cdot\mathbf{n}+(\mathbf{a}\cdot\mathbf{n})\wedge\mathbf{n}+(\mathbf{a}\cdot\mathbf{n})\cdot\mathbf{n}=\\ &= 0+(\mathbf{a}\wedge\mathbf{n})\cdot\mathbf{n}+(\mathbf{a}\cdot\mathbf{n})\mathbf{n}+\mathbf{0}\end{aligned}$$

The last non-null term is the projection of **a** on **n**, and the other term must be, therefore, the perpendicular one. This can be made explicit as follows:

$$(\mathbf{a}\wedge\mathbf{n})\cdot\mathbf{n}=-\mathbf{n}\cdot(\mathbf{a}\wedge\mathbf{n})=\mathbf{a}-(\mathbf{n}\cdot\mathbf{a})\mathbf{n}\,.$$

If $\mathbf{B}$ is, for instance, a bivector, which thus defines a plane, we similarly have

$$\begin{aligned}(\mathbf{aB})\mathbf{B}^{-1} &= (\mathbf{aB})\frac{\mathbf{B}}{B^2} = (\mathbf{aN})\mathbf{N} = \\ &= (\mathbf{a}\wedge\mathbf{N})\mathbf{N}+(\mathbf{a}\cdot\mathbf{N})\mathbf{N} = \mathbf{a}_\perp + \mathbf{a}_{\shortparallel}\end{aligned}$$

where the relation of $\mathbf{N}$ to $\mathbf{B}$ is obvious. Readers should try to figure out why one need not decompose $(\mathbf{a}\wedge\mathbf{N})\mathbf{N}$ and $(\mathbf{a}\cdot\mathbf{N})\mathbf{N}$ further. Similar considerations apply to the decomposition of a vector into components.

The preceding considerations apply equally to the decomposition of a vector into complementary subspaces of dimensions r and $n-r$. Of course, we would need a little bit more algebraic development, but the general line of the argument should be clear by now. Another example of almost computing in autopilot is the treatment of reflections and rotations. Even explicit rotations in three dimensions are very easy to handle, without the need for Euler angles. But the main reason to study Clifford algebra is the calculus that one can develop based on it. That will be done in the section on Kähler. The section on Cartan is an intermediate step to get there.

5 Cartan

5.1 Exterior calculus for scalar-valuedness

The exterior calculus makes use of exterior products only (specifically of the differential of the coordinates), rather than exterior products of (so called) tangent vectors, like $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}, \mathbf{l}$...We can view the exterior product as contained in the Clifford product. Correspondingly, one can view Cartan's exterior calculus as contained in the Kähler calculus,.the latter requiring that the manifold be endowed with a metric. At this point, we consider only exterior products and, correspondingly, the Cartan calculus on its own. It already exists on differentiable manifolds not endowed with a metric and, thus, not endowed either with a Kähler calculus.

The exterior product of differential 1-forms satisfies the equation

$$dx^\mu \wedge dx^\nu + dx^\nu \wedge dx^\mu = 0.$$

The non-null exterior products of a number r of 1-forms are called differential r-forms.

We introduce the operator d for exterior differentiation. It increases the grade of differential forms by a unit; if α is a differential r-form of grade r, $d\alpha$ is a differential $(r+1)$-form. It is called the exterior derivative of α, though the name exterior differential would be more appropriate. We shall define it in the next subsection. For motivational purposes, we alter the logical order and proceed to give the greatest theorem of the exterior calculus of scalar-valued differential forms. It states that

$$\int_{\partial R} \alpha = \int_R d\alpha,$$

where R is an $(r+1)$-"surface", and ∂R is its boundary. This is called the general Stokes theorem. It comprises , among others, the theorems of Gauss and Stokes, but it applies to spaces of arbitrary dimension and concerns also domains of integration of greater dimensionality.

Suppose that α is such that $d\alpha$ is zero. The general Stokes theorem becomes

$$\int_{\partial R} \alpha = 0,$$

which is a compact statement comprising multiple conservation laws, depending on how one splits the boundary, and on the signature of the metric and on the dimension of the manifold (read space), and on the grade of α. For example, if the signature is Lorentzian, the conservation law is used in the form that integrals over the spatial subspace take the same value at different instants of time. If β is a differential 2-form such that $d\beta = 0$, and if the signature of the metric is positive definite, the flux of β through a closed surface is zero. For instance, in the exterior calculus the magnetic field is a differential 2-form in 3-space. This annulment amounts to the statement that the flux of β through a closed surface is zero

5.2 Exterior differentiation: definition and properties

A simple example will suffice to show how one proceeds to exterior differentiate. Given

$$\alpha = a_{\mu\nu\lambda} dx^\mu \wedge dx^\nu \wedge dx^\lambda,$$

$d\alpha$ is given by

$$d\alpha = da_{\mu\nu\lambda} \wedge dx^\mu \wedge dx^\nu \wedge dx^\lambda,$$

and, therefore,

$$d\alpha = \frac{\partial a_{\mu\nu\lambda}}{\partial x^\rho} dx^\rho \wedge dx^\mu \wedge dx^\nu \wedge dx^\lambda.$$

This definition is independent of whether we are in Cartesian coordinates or not, and applies in general spaces known as differentiable manifolds. Exterior differentiation does not require the manifold to have any structure other than being differentiable, thus not a metric, and not a connection.

Important properties are that

$$d(\alpha + \beta) = d\alpha + d\beta$$

and that, if α and β are of respective grades r and s, we have:

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta \, + \, (-1)^r \alpha \wedge d\beta$$

(the grade of the second factor does not matter because d acts from the left). Of great importance is the property of d that its square is zero, $dd\alpha = 0$ (if α is scalar-valued). Also, if $d\alpha = 0$, there is a β such that $\alpha = d\beta$, though this result is not necessarily valid globally. Think of the integration of $d\phi$ on the plain, where ϕ is angle. We have $d(d\phi) = 0$. Integrating clockwise and counterclockwise, we get to the same point in the plane with values for ϕ that differ by 2π.

We write the fundamental theorem of the calculus as

$$\int_\gamma df = f(B) - f(A),$$

where γ is the interval between points A and B. In other words, the ordered pair (A, B) is the oriented boundary $\partial\gamma$ of the segment γ. In view of Stokes generalized theorem, this right hand side may be viewed as

$$\int_\gamma df \quad \left(= \int_{\partial\gamma} f \right).$$

Everything falls elegantly in place.

5.3 Exterior calculus for vector-valuedness

We use polar coordinates in the Euclidean plane to get a feeling for what is called Cartan's method of the moving frame [15]. Using polar coordinates and associated fields of bases (also known as frame fields when using this method), we have

$$\mathbf{e}_\rho = \mathbf{i}\cos\phi + \mathbf{j}\sin\phi, \qquad \mathbf{e}_\phi = -\mathbf{i}\sin\phi + \mathbf{j}\cos\phi$$

We define connection forms by means of

$$\begin{aligned} d\hat{\mathbf{e}}_\rho &= -\mathbf{i}\sin\phi \; d\phi + \mathbf{j}\cos\phi \; d\phi \equiv \omega^\rho_\rho \hat{\mathbf{e}}_\rho + \omega^\phi_\rho \hat{\mathbf{e}}_\phi, \\ d\hat{\mathbf{e}}_\phi &= -\mathbf{i}\cos\phi \; d\phi + \mathbf{j}\sin\phi \; d\phi \equiv \omega^\rho_\phi \hat{\mathbf{e}}_\rho + \omega^\phi_\phi \hat{\mathbf{e}}_\phi, \end{aligned}$$

where we have used the standard assumption that $\mathbf{i}$ and $\mathbf{j}$ are constant vector fields, and where the indices ρ and ϕ play the role of names rather than of dummy indices that one sums over. We have used the circumflex hats to indicate that this is an orthonormal frame field. For such frame fields, the ω^ρ_ρ and ω^ϕ_ϕ happen to be zero. The ω's that satisfy those equations represent the so called connection of Euclidean space in terms of the particular frame field used. The ω coefficients in the equations for

$(d\mathbf{i}, d\mathbf{j})$ in terms of the basis $(\mathbf{i}, \mathbf{j})$ are all zero. The connection does, therefore, depend on frame field. In compact form:

$$\hat{\mathbf{e}}_\mu(P + dP) - \hat{\mathbf{e}}_\mu(P) = d\hat{\mathbf{e}}_\mu = \omega_\mu^\nu \hat{\mathbf{e}}_\nu,$$

where P represents the points on the manifold, and with summation over the dummy repeated index μ. These equations exhibit the geometric significance of $d\hat{\mathbf{e}}_\mu$ and constitute what Cartan calls the exterior differentiation of a field of vector bases. They give in differential form (meaning that they have to be integrated) how the basis changes as a function of point. We shall follow Cartan and Kähler to refer to the d differentiation of any vector-valued differential form, and to vector fields in particular, as exterior differentiation. In this way, we do not need the terms covariant and exterior covariant, as the specifics will be determined by the nature of the object to which d is applied.

On arbitrary manifolds, we cannot express a field of vector bases in terms of a fixed basis, as a given differentiable manifold might not be endowed with a constant basis field. At the same time, the symbol d in $d\hat{\mathbf{e}}_\mu$ and in $d\mathbf{v}$ does not mean differentiation of $\hat{\mathbf{e}}_\mu$ and $\mathbf{v}$, which are not vector valued functions in some vector space, but different tangent vector spaces. Just think of the planes tangent to a sphere. These vector spaces may or may not be identifiable, depending on connection. On differential manifolds of arbitrary dimension, ω_μ^ν is introduced either ab initio or in some other way. An example of ab initio introduction is the "Christopher Columbus connection". Before sailing deep into the Atlantic Ocean in 1492, Columbus instructed the captains of the two other ships that accompanied him: "keep always the same direction, West". He was implying that the parallels of the surface of the earth (the pole excluded to avoid divergences of concepts like torsion) were to be considered as lines of constant direction. Modernly, we would state that

$$d\hat{\mathbf{e}}_\theta = d\hat{\mathbf{e}}_\phi = 0, \qquad \text{or } \omega_\mu^\nu = 0,$$

where $\hat{\mathbf{e}}_\theta$ and $\hat{\mathbf{e}}_\phi$ are unit tangent vectors in the direction of the parallels and meridians. In other words, $(\hat{\mathbf{e}}_\theta, \hat{\mathbf{e}}_\phi)$ constitute a constant basis field. It is special in that regard. This connection was not explicit in the mathematical literature until a paper by Cartan in1924.

Another connection on the surface of the sphere —one which does not accept a constant basis field— was given by Levi-Civita in 1917. It is called the canonical connection of the metric, but we shall not enter into why this is so. Cartan showed how to compute so called metric compatible affine connections rather easily. For the Levi-Civita connection, which is one of them, his technique readily yields

$$d\hat{\mathbf{e}}_\theta = \cos\theta \; d\phi \; \hat{\mathbf{e}}_\phi, \qquad d\hat{\mathbf{e}}_\phi = -\cos\theta \; d\phi \; \hat{\mathbf{e}}_\theta,$$

from which one can read the connection differential forms.

In Euclidean space, the exterior differential of a vector field is given by

$$d\mathbf{v} = d(v^\mu \hat{\mathbf{e}}_\mu) = (dv^\mu)\hat{\mathbf{e}}_\mu + v^\mu(d\hat{\mathbf{e}}_\mu) = (dv^\nu)\hat{\mathbf{e}}_\nu + v^\mu \omega_\mu^\nu \hat{\mathbf{e}}_\nu.$$

All the members in this set of equations also make sense on arbitrary manifolds, provided that, as we already said, one does not take $d\hat{\mathbf{e}}_\mu$ to be the result of differentiating any basis field. It is rather introduced ab initio as a rule to compare vectors at a distance. Or it is obtained from some other condition. In one case as in the other, it constitutes the input in a differential system that has to be integrated, if integrable, or that one has to integrate on specific curves, if not integrable.

With the foregoing proviso about the concept of differentiation, we differentiate $d\mathbf{v}$. We use the Leibniz rule for differential forms as per the previous subsection and take into account that vector fields are differential 0-forms. We get five terms, corresponding to the five factors in $(dv^\nu)\mathbf{e}_\nu + v^\mu \omega_\mu^\nu \mathbf{e}_\nu$. The first term is zero. the second and third ones cancel each other out. From the last two terms, we obtain

$$dd\mathbf{v} = dd(v^\mu \hat{\mathbf{e}}_\mu) = v^\mu(d\omega_\mu^\nu - \omega_\mu^\lambda \wedge \omega_\lambda^\nu)\hat{\mathbf{e}}_\nu.$$

In the particular case of Euclidean space, $d\omega_\mu^\nu - \omega_\mu^\lambda \wedge \omega_\lambda^\nu$ is zero, independently of frame field, even if the ω_μ^ν are zero only in special frame fields. It is called the affine curvature (Just compute $dd\mathbf{v}$ using a constant basis field $\mathbf{a}_i$).However, what happens in Euclidean space may be used to understand of objects which, like the curvature, are not zero in generalizations of Euclidean geometry. We cannot deal here with such issues, but let us say that, for example, that the curvature differential 2-form, $d\omega_i^j - \omega_i^k \wedge \omega_k^j$ is Lie algebra valued. This is precisely valuedness in the Lie algebra of groups that acts on the frames of Euclidean space.

As important as the affine curvature is the torsion, which is the formal exterior differential of $d\mathbf{P}\,(= \omega^\nu \hat{\mathbf{e}}_\nu)$, where ω^ν.is a linear combination of the differentials of the coordinates and which forms a basis of differential 1-forms. In the case of Euclidean plane, we have, for example,

$$\omega^1 = d\rho, \qquad \omega^2 = \rho\, d\phi.$$

The formal exterior derivative of $d\mathbf{P}$ is the torsion.

The ω_μ^ν are a linear combination of the ω^λ

$$\omega_\mu^\nu = \Gamma_{\mu\lambda}^\nu \omega^\lambda,$$

and, therefore,

$$d\omega_\mu^\nu = \Gamma_{\mu\lambda/\theta}^\nu \omega^\theta \wedge \omega^\lambda = (\Gamma_{\mu\lambda/\theta}^\nu - \Gamma_{\mu\lambda/\theta}^\nu)(\omega^\theta \wedge \omega^\lambda),$$

and

$$\omega_\mu^\pi \wedge \omega_\pi^\nu = (\Gamma_{\mu\theta}^\pi \Gamma_{\pi\nu}^\lambda - \Gamma_{\mu\lambda}^\pi \Gamma_{\pi\theta}^\nu)(\omega^\theta \wedge \omega^\lambda)$$

where the slash bar as a subscript means that the differentials of the Γ's are expressed in terms of the basis ω^θ rather than dx^θ, and where the parenthesis around $\omega^\theta \wedge \omega^\lambda$ indicates that $\theta < \lambda$. After substitution of these expressions for $d\omega_i^j$ and $\omega_i^m \wedge \omega_m^j$ in $d\omega_i^j - \omega_i^k \wedge \omega_k^j$, the coefficients of $(\omega^\theta \wedge \omega^\lambda)$ constitute the coefficients of the curvature.

We are not providing here the rigorous definition and/or introduction of the concepts that we are using. We are simply giving the flavor of the moving frame method of Cartan. A more detailed and rigorous presentation can be found in one of our books to be published in 2010 [16]. As in Cartan's original presentation, the book makes use of differential forms (no tensors, except of the occasional tensor-valuedness of the differential forms) and moving frames, with focus on integrability issues, on the study of affine space before affine connections, on Cartan's generalization of Klein's program and on Lie algebra valuedness of connection and affine curvature. Several simple examples of manifolds endowed with torsion will also be considered.

6 Kähler

6.1 Kähler calculus for scalar-valuedness

For simplicity, we shall restrict ourselves to Euclidean (and pseudo-Euclidean) spaces in terms of Cartesian (respectively pseudo-Cartesian) coordinates. The Kähler calculus is based on the Kähler algebra, term used to refer to the Clifford algebra of differential forms. In spacetime, it is defined by the relation

$$dx^\mu \vee dx^\nu + dx^\nu \vee dx^\mu = 2\eta^{\mu\nu},$$

where $\eta^{\mu\nu}$ constitutes the elements of the diagonal matrix $(1, -1, -1, -1)$. This expression is analogous to

$$\mathbf{a} \cdot \mathbf{b} = \frac{1}{2}(\mathbf{a} \vee \mathbf{b} + \mathbf{b} \vee \mathbf{a}).$$

As a ready consequence of the relation between exterior and Clifford products, it contains the defining relation $dx^\mu \wedge dx^\nu + dx^\nu \wedge dx^\mu = 0$ for the exterior algebra of differential 1-forms. Because there is no dot product in the exterior algebra, it cannot deal with vector identities for expressions such as

$$\nabla \times (\mathbf{a} \times \mathbf{b}), \quad \nabla \times (\nabla \times \mathbf{a}), \quad \nabla(\mathbf{a} \cdot \mathbf{b}).$$

There are corresponding limitations in the Cartan calculus.

Let u be a differential form. The equations that follow expose the structure of differentiations in the Kähler calculus. The Kähler derivative, ∂, is the sum of the exterior, d, and interior, δ, derivatives:

$$\begin{aligned} \partial u &= dx^\mu \vee d_\mu u = du + \delta u \\ du &= dx^\mu \wedge d_\mu u, \quad \delta u = dx^\mu \cdot d_\mu u \end{aligned}$$

This reflects the relationship between the different products involved in the underlying algebra.

In terms of Cartesian coordinates, we have

$$d_\mu u = a_{\nu\lambda\rho},_\mu \, dx^\nu \wedge dx^\lambda \wedge dx^\rho,$$

from which the known expression for the exterior derivative follows:

$$du = a_{\nu\lambda\rho},_\mu \, dx^\mu \wedge dx^\nu \wedge dx^\lambda \wedge dx^\rho.$$

Both $d_\mu u$ and δu, depend on the connection. These formulas are simple because of our use of Cartesian coordinates (these do not exist on arbitrary manifolds!). For a differential 1-form, we have

$$d_\mu(a_\nu dx^\nu) = a_\nu,_\mu \, dx^\nu$$

and, therefore,

$$\delta u = dx^\mu \cdot d_\mu(a_\nu dx^\nu) = dx^\mu \cdot a_\nu,_\mu \, dx^\nu = a^\mu,_\mu \,,$$

which is nothing but the divergence. Knowledgeable readers will identify the interior derivative with the coderivative, but this is only if the connection is Levi-Civita's, as is the case in Euclidean space.

6.2 The conservation law in the Kähler calculus

Conservation laws are specific developments of the statement that the exterior derivative of some differential form is zero. There is, however, a theorem relating the exterior derivative to quadratic expressions involving the Kähler derivative. It has great importance for quantum mechanics. For any two differential forms, whether homogeneous or not, one has

$$d(u, v)_1 = (u, \partial v) + (v, \partial u),$$

where products $(\_,\_)$ and $(\_,\_)_1$ denote respective differential n-forms and $(n-1)$-forms that one builds with the two differential forms inside each parenthesis. We do not need to know details for present purposes. Suffice to say that these products depend on the metric. This theorem is in reality a collection of theorems since one can, say, replace u or v or both with anything we want, say $u\partial\partial v$, etc. The great value of this theorem is that, whenever we manage to have differential forms such that the right hand side is zero,

$$(u, \partial v) + (v, \partial u) = 0,$$

so is the left hand side,

$$d(u, v)_1 = 0.$$

In other words, $(u, v)_1$ is a conserved differential form. This road to conservation will prove its worth in quantum mechanics.

6.3 Kähler's quantum mechanics defined

At its most basic, quantum mechanics may be viewed as the theory of the Kähler equation and its offsprings (Schrödinger, Pauli). This implies that, if physicists had not done so already, the mathematics of relativistic quantum mechanics would have been discovered by mathematicians, as a step in the evolution of the calculus,

The characterization of quantum mechanics in terms of the Kähler calculus is: a field theory where the differential form for a field is its own source. The equations that would respond to this characterization in a theory based in just one ordinary function would be, in order of increasing complexity: $f' = 0$, $f' = f$ and $f' = af$... The parallel equations in the present calculus are

$$\partial u = 0, \quad \partial u = u, \quad \partial u = au.$$

The last of these is the Kähler version of the Dirac equation. Actually $\partial u = au$ reproduces anything that the Dirac equation can do, and more and better. For the electromagnetic coupling, and up to universal constants, Kähler proposed

$$a = m + eA.$$

Thus Kähler's equation with EM coupling becomes

$$\partial u = (m + eA) \vee u.$$

One can define a "conjugate Kähler equation", whose solutions, v, are closely related to those, u, of the original equation. The following then is the case: $(u, \partial v) + (v, \partial u) = 0$. We thus have

$$d(u, v)_1 = 0.$$

Let η be the operator that reverses the sign of all the factors in a product of differential 1-forms. Let overbar denote complex conjugate. For EM coupling, the $\eta\bar{u}$ are solutions of the conjugate equation. Thus, if u and u' are solutions of the original Kähler equation, one has $d(u, \eta\bar{u}')_1 = 0$ and, in particular,

$$d(u, \eta\bar{u})_1 = 0.$$

This yields the conservation of charge (see below), accompanied of a rich load of information. We shall not get into that because of the technicalities involved. But we shall speak of them in general terms in order to get the important result about the sign of the energy of antiparticles.

6.4 Spinors, charge and particles/antiparticles

Kähler was a great expert on the theory of exterior differential systems, a credential that brings high relevance to his quantum mechanical work. Indeed, any differential equation or system thereof can be written as a so called exterior system, i.e expressing relations between exterior differential forms and their derivatives. Kähler showed the appropriate way of solving such systems in the spacetime case, more specifically he enriched our knowledge of the structure of solutions of such systems that are "proper functions". He revealed a wealth of information that is lost if one approaches the same system but written in alternative ways ("component equations"). The concept of spinor as an element of relevant ideals in the algebra emerges in the process. Related to the spinors in the case of electromagnetic coupling, energy emerges in association with charge, and chirality in association with spin. The expression for spin itself is the result of an appropriate treatment of the rotation operator on differential forms.

It transpires from this description that quantum mechanics might be conceived with a lesser emphasis on probability densities and spinors, and more on the field itself without regard for probability amplitudes. It is not that spinors and probability densities are not important. It simply happens that spinor fields are derived concepts, and so are particles and thus the probabilities to find them for a given "primordial field configuration". At this point we would not be very justified in using the term primordial. We have connected elsewhere, [17],[18], the Kähler equation for the primordial field with the equations of structure in differential geometry.

6.4.1 Spinors

Let us see for example the emergence of the concept of spinor. It is based on the concept of constant differential, i.e. differential forms such that

$$d_\mu c = 0 \qquad for\ all\ i,$$

It then follows that

$$\partial(u \vee c) = a \vee (u \vee c),$$

The constant differentials

$$\epsilon^{\pm} = \frac{1 \mp dt}{2}, \qquad \tau^{\pm} = \frac{1 \pm idx^1 \vee dx^2}{2}$$

are idempotents (an idempotent is anything which is equal to its square). The product of the two forms ϵ yields zero. Their sum is unity. Parallel statements apply to the τ's. The ϵ's commute with the τ's. The solution u in the presence of time translational and rotational symmetries for given value of intrinsic angular momentum and energy can be written as a sum of four independent solutions of the type

$$u = e^{iM\phi - iEt/\hbar} \ {}^{\pm}q^* \vee \epsilon^{\pm} \vee \tau^*,$$

where M can be shown to be an integer in the actual solving of the Kähler equation and where q depends exclusively on the ρ and z coordinates and their differentials. Each of these four independent solutions correspond to Dirac spinors. This represents an enrichment of the concept of proper functions of energy and momentum.

6.4.2 Charge

In terms of ideals associated exclusively with time translation, we can decompose the proper wave for a given value of the energy as

$$u = e^{-iEt/\hbar}({}^{+}u \vee \epsilon^{+} + \ {}^{-}u \vee \epsilon^{-}),$$

where the superscript indicates the sign of the charge component of the solution u of the Kähler equation. Using this decomposition in the expression for the conserved differential form $(u, \eta\bar{u})_1$ and performing a space and time split (in other words, a separation of space differentials from time differentials) led him, in the case of electromagnetic coupling to

$$\begin{aligned}(u, \eta\bar{u})_1 \quad = \quad & \frac{1}{2}\{{}^{+}u, {}^{+}\bar{u}\} + \frac{1}{2}\{{}^{+}u, \eta^{+}\bar{u}\}_1 \vee idt, \\ & -\frac{1}{2}\{{}^{-}u, {}^{-}\bar{u}\} + \frac{1}{2}\{{}^{-}u, \eta^{-}\bar{u}\}_1 \vee idt\end{aligned}$$

where $\{\ ,\ \}$ and $\{\ ,\ \}_1$ are the same products as $(\ ,\ )$ and $(\ ,\ )_1$ except that they now pertain to 3-space rather than spacetime, and are thus computed with the Euclidean metric. This takes the form of the conserved 3-form current, which is $\rho w - j^i dx^j \wedge dx^k \wedge dt$. We thus see that the density to which this conserved current refers is contained in

$$\frac{1}{2}\{{}^{+}u, {}^{+}\bar{u}\} - \frac{1}{2}\{{}^{-}u, {}^{-}\bar{u}\}.$$

The characterization of the state of positron by $(u \vee \epsilon^{+} = u,\ u \vee \epsilon^{=} = 0)$, and of the electron by $(u \vee \epsilon^{-} = u,\ u \vee \epsilon^{+} = 0)$, brings about the interpretation that there are contributions of opposite signs to that density. It becomes either one contribution or the other in the cases of pure electron $(u = {}^{-}u)$ and pure positron $(u = {}^{+}u)$. This leads to the association of the density 3-form for electrons and positrons as

$${}^{\mp}\rho w = \mp\frac{|e|}{2}\{{}^{\mp}u, {}^{\mp}\bar{u}\},$$

where e is some proportionality constant retrospectively identified with the charge of the electron (or proportional to it with a positive proportionality constant). We have taken into account that one can prove that both $\{{}^{\mp}u, {}^{\mp}\bar{u}\}$ are non negative under the conditions (like positive definite metric) of the physical problem in question. The emergence of, and association with negative energies of antiparticles has thus to be viewed as an artifact of the Dirac theory, which is one of the reasons why we speak of its being superseded by the Kähler theory. All this was done by Kähler half a century ago

6.4.3 Emergence of generalized momentum

The present author has recently elaborated further in making the case that the Kähler calculus should be the calculus for physics in general and quantum mechanics in particular [5]. A summary follows. The Kähler equation with minimal EM coupling is

$$-i\hbar\partial u = \frac{1}{c}(imc^2 - ec\phi dt + eA_i dx^i) \vee u,$$

where ϕ now is the time component of the potential 1-form. Since the mass term dominates at low energies, we rewrite the wave function of the electron as

$$u = e^{-imc^2 t/\hbar} R(x, dx, t) \vee \epsilon^-.$$

R depending slowly on time. All the dependence on dt is, of course contained in ϵ^-, since dt is a linear combination of ϵ^- and ϵ^+, and we are dealing with electrons. Take into account that $\partial u = dt \vee u_{,t} + dx^i \vee u_{,i}$.and define P as

$$P \equiv dx^j(-i\partial_j - eA_j).$$

One obtains "the "master system"

$$R_{,t} = -P \vee \eta R - ie\phi R + im(R - \eta R).$$

Further development yields

$$\begin{aligned} \varphi_{,t} &= P \vee \chi - ie\phi\varphi, \\ \chi_{,t} &= -P \vee \varphi - ie\phi\chi + 2im\chi, \end{aligned}$$

where

$$\varphi \equiv \frac{1}{2}(R + \eta R) \qquad \chi \equiv \frac{1}{2}(R - \eta R).$$

6.4.4 Pauli and Schrödinger's equations

We may now develop the master system further. In a first approximation, we have

$$\chi_1 \equiv -\frac{i}{2m} P \vee \varphi,$$

and

$$i\varphi_{,t} = \frac{1}{2m} P \vee (P \vee \varphi) + e\phi\varphi.$$

which in turn yields

$$i\varphi_{,t} = \frac{1}{2m} P^2\varphi + \frac{ie}{2m} B_k dx^i \vee dx^j \vee \varphi + e\phi\varphi.$$

This is the Pauli equation in terms of differential forms. The Schrödinger equation,

$$i\varphi_{,t} = \frac{1}{2m} p^2\varphi + e\phi\varphi,$$

readily emerges if the magnetic field is zero. We do not go any further as the foregoing is enough for our illustration. Readers interested in the next approximation and other important results related to antiparticles are referred to [5].

7 Concluding remarks

We hope to have shown in some cases and intimated in other cases the enormous advantage for mathematics and physics that follows from the use of a richly endowed calculus of differential forms. Scalar-valued differential forms were involved for some applications and vector-valued ones for other applications. Because of the limited scope of this paper, vector-valued differential forms have been considered only in the Cartan context, i.e. with the exterior calculus. In other words, we have used the Kähler calculus only for scalar-valued differential forms. Owing to the same limitation, we have not dealt with Lie algebra-valued differential forms, Finslerian and Kaluza-Klein contexts, etc. All that speaks of the tremendous potential for applications that remains in this calculus. These additional developments should create a still higher mathematical platform for physics, which this author is committed to build [10]. It will only take climbing to the shoulders of the mathematical giants that we have mentioned in order to see what nobody has yet seen. The purpose of this paper will have been achieved if some readers are hereby prompted to join the climb.

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ДИФФЕРЕНЦИАЛЬНЫЕ ФОРМЫ: ОТ КЛИФФОРДА ЧЕРЕЗ КАРТАНА К КЭЛЕРУ.

Джозе Г. Варгас

Университет Северной Каролины, Колумбия, Северная Каролина, США
josegvargas@earthlink.net

Продемонстрированы пределы векторного, тензорного и спинорного дираковского исчислений для мотивации введения кэлерова исчисления интеграндов, заменяющего все три вышеперечисленные. При этом кэлерово исчисление вводится в три этапа.

Во-первых, мы формулируем основы алгебры Клиффорда, лежащей в основе кэлерова исчисления, и пригодной как для евклидовых, так и для псевдоевклидовых векторных пространств любого числа измерений. Показано, что обычная векторная алгебра представляет собой "поврежденную" алгебру Клиффорда, причем "повреждение" рассматриваемого типа возможно лишь в 3-мерном векторном пространстве. Клиффордово произведение строится как сумма внешнего и внутреннего произведений, если, по крайней мере, один из сомножителей является вектором. Грубо говоря, эти произведения обобщают обычные векторное и скалярное произведения и включают в себя внешнюю алгебру.

В качестве промежуточного шага на пути к исчислению Кэлера мы кратко формулируем основы исчисления внешних скалярно-значных дифференциальных форм Картана, рассматриваемых здесь как обычные скалярно-значные подынтегральные выражения в кратных интегралах. Мы также делаем небольшой экскурс в исчисление внешних векторно-значных дифференциальных форм, реализующих метод подвижного репера в дифференциальной геометрии.

Далее мы представляем основы исчисления дифференциальных форм Кэлера. Оно относится к внешнему исчислению так же, как алгебра Клиффорда относится к внешней алгебре. Ввиду ограничений по объему и сложности, мы останавливаемся лишь на скалярно-значных дифференциальных формах, что вполне достаточно для приложений в области релятивистской квантовой механики с электромагнитным взаимодействием. Использование исчисления Кэлера не приводит к решениям с отрицательными энергиями.

Ключевые слова: дифференциальные формы, гиперкомплексные числа, алгебра Клиффорда, внешнее исчисление, подвижный репер, исчисление Кэлера.