SPECTRAL PROPERTIES AND APPLICATIONS OF THE NUMERICAL MULTILINEAR ALGEBRA OF M-ROOT STRUCTURES

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In the framework of supersymmetric tensors and multivariate homogeneous polynomials, the talk discusses the 4-th order Berwald-Moor case. The eigenvalues and eigenvectors are determined; the recession and degeneracy vectors, characterization points, rank, asymptotic rays, base index, are studied. As well, the best rank-one approximation is derived, relations to the Berwald-Moor poly-angles are pointed out, and a brief outlook on real-world applications is provided.

MSC: 65F30, 15A18, 15A69.

Keywords: recession vector, degeneracy vector, characterization points, asymptotic rays, best rank-one approximation.

1 Introduction

Within the framework of applied computational mathematics, the numerical multilinear algebra has developed in recent years in connection with computational topics regarding higher-order tensors, as: tensor decomposition, computation of tensor rank, eigenvalues and eigenvectors, lower-rank approximation of tensors, numerical stability and perturbation analysis of tensor computation, etc. (e.g., [15], [14]).

The wide applications of this field include digital image restoration, psycho-, chemo- and econnometrics, multi-way data analysis, blind source separation and blind source deconvolution in signal processing, higher-order statistics, etc (e.g., [1–6, 9, 16, 17]).

As example, the blind source separation problem (BSS problem, known as Independent component analysis – ICA problem) from signal processing has, among multiple aims, both interception and classification in military applications and surveillence of communications in the civil resort. Here, the problem of blind separation of convolutive mixtures is known as Blind Deconvolution (BD) - problem.

This problem attempts in separating and recovering statistically an independent stochastic process (a source vector $x \in \mathbb{R}^n$) given the output $z = Ax + v$, where $z \in \mathbb{R}^m$ is the observed random variable of dimension m, A is an $m \times n$ mixing matrix and $v \in \mathbb{R}^m$ is the $x-$independent noise vector. The main method to recover the source is to project the observed variable z onto a vector u selected such that the inner product $\langle u, z\rangle$ is maximized. The problem reduces ultimately to finding the best approximation to a 4-th cumulant supersymmetric tensor T by another rank-one tensor – where in case of convexity/concavity assumptions, the symmetric higher-order power method converges.

In the present work, using the tools developed in [14, 15], we investigate spectral aspects – including the 1-rank approximantion problem, of the Berwald-Moor supersymmetric $(0, 4)-$tensor:

$$A_{ijkl} = \begin{cases} \dfrac{1}{24!}\,, & \text{for } \{i,j,k,l\} = \{1,2,3,4\} \\ 0, & \text{otherwise.} \end{cases} \tag{1.1}$$

2 Eigenvalues and eigenvectors of supersymmetric tensors. The Berwald-Moor case

In general, for a supersymmetric tensor $A \in \mathcal{T}^0_m(\mathbb{R}^n)$ of order m on $\mathbb{R}^n$, we say that $\lambda \in \mathbb{R}$ is an (Z-)eigenvalue and a vector $y \in \mathcal{T}^1_0(\mathbb{R}^n) \equiv \mathbb{R}^n$ is an associated (Z-)eigenvector, if they satisfy the system:

$$Ay^{m-1} = \lambda y; \quad g(y,y) = 1, \tag{2.1}$$

where we have considered the transvection

$$Ay^{m-1} = C^1_1 C^2_2 \dots C^{m-1}_{m-1}(A \otimes \underbrace{y \otimes \dots \otimes y}_{m-1 \text{ times}}),$$

C^i_j is the transvection operator on the corresponding indices and $y = y^i e_i$ is the Liouville vector field on the flat manifold $\mathbb{R}^n$, considered at some point $x \in \mathbb{R}^n$, and $g = \delta_{ij} dx^i \otimes dx^j$ is the flat Euclidean metric on $\mathbb{R}^n$ which implicitly raises/lowers the tensor indices. For the complex variant, λ and y are simply called *eigenvalue* and *eigenvector*, respectively.

As a corollary to [14, Theorem 1, p.1312], after tedious computations, one infers the following result:

Theorem 2.1 *Consider the Berwald-Moor supersymmetric tensor A given in (1.1). Then*

a) The eigenvalues of A are $\lambda \in \sigma(A) = \{0, \pm\frac{1}{16}\}$, with the associated eigenvectors

$$\begin{cases} S_{\lambda=0} & = \{(x_1,x_2,x_3,x_4) \mid \exists i<j; i,j \in \overline{1,4}, \exists\theta \in [0,2\pi), \\ & \qquad x_i = \cos\theta, x_j = \sin\theta, x_k = 0, \forall k \in \overline{1,4} \setminus \{i,j\}\} \\ S_{\lambda=1/16} & = \{(x_1,x_2,x_3,x_4) \mid x_1,x_2,x_3,x_4 \in \{\pm 1/2\}, x_1x_2x_3x_4 > 0\} \\ S_{\lambda=-1/16} & = \{(x_1,x_2,x_3,x_4) \mid x_1,x_2,x_3,x_4 \in \{\pm 1/2\}, x_1x_2x_3x_4 < 0\} \end{cases}$$

b) The supersymmetric tensor A is not positive semi-definite.

c) The hyperquartic surface $\Sigma: \ Ay^4 = c$ for $c > 0$ is nonempty and the distance $d = dist(O,\Sigma) = 2\sqrt[4]{c}$ which occurs at the $2 + \binom{4}{2} = 8$ points $y_* \cdot d$, given by the directions

$$y_* \in \left\{ \frac{1}{2}(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4) \mid \varepsilon_{1,2,3,4} \in \{\pm 1\}, \varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4 = 1 \right\}. \tag{2.2}$$

d) Σ does not surround a bounded region in $\mathbb{R}^4$.

Proof. a) Tedious computation for solving $Ay^3 = \lambda y, g(y,y) = 1$. b) Apply [14] for $\sigma(A) \not\subset \mathbb{R}_+$. c) $\sigma(A) \cup \mathbb{R}^*_+ \neq \emptyset$ and the minimal distance $d = d(O,\Sigma) = (c/\lambda_{max})^{1/m}$ occurs at $d \cdot y_*$. d) is a consequence of non-positive semidefiniteness and [14].

Remark. The eigenvalues considered above are the $Z-$eigenvalues studied by L. Qi in [13].

Alternatively, L. Qi has defined the following spectral objects ([12,15]):

Definition. A real number $\lambda \in \mathbb{R}$ is an (H-)eigenvalue and a vector $y \in \mathbb{R}^n$ is an associated (H-)eigenvector, if they satisfy the homogeneous polynomial system of order $n-1$:

$$(Ay^{m-1})_k = \lambda (y_k)^{m-1}. \tag{2.3}$$

For the complex variant, λ and y are simply called *(E-)eigenvalue* and *(E-)eigenvector*, respectively.

Regarding the existence of eigenvalues/eigenvectors, the following result holds true:

Theorem 2.2 *$H-$eigenvalues and $Z-$eigenvalues always exist for even supersymmetric tensors. A supersymmetric tensor A is positive definite/semidefinite iff all its $H-$ (or $Z-$) eigenvalues are positive/non-negative.*

3 Recession vectors and rank

Among vectors which characterize a supersymmetric tensor $A \in T^0_m(\mathbb{R}^n) \equiv (T^0_m)_x(\mathbb{R}^m)$, we have the following:

Definitions. a) A recession vector of A is $y \in V = \mathbb{R}^n$ such that for all $x \in V$ and $\alpha \in \mathbb{R}$, we have

$$A(x + \alpha y)^m = Ax^m,$$

i.e., the multi-action of A on any line passing through any point x which is oriented by y, depends only on the direction of y (i.e., A has the same multi-action on the whole pencil of parallel lines of direction y).

b) The mixed $(m-k, k)$ action of A onto the pair $(u, v) \in \mathbb{R}^n \times \mathbb{R}^n$ is given by

$$Ax^{m-k}y^k = C^1_1 \dots C^m_m A \otimes \underbrace{u \otimes \cdots \otimes u}_{m-k \text{ times}} \otimes \underbrace{v \otimes \cdots \otimes v}_{k \text{ times}}.$$

We note that due to the supersymmetry we have ([14]) $Ax^{m-k}y^k = Ay^kx^{m-k}$ and $A(x + \alpha y)^m = \sum_{k=0}^{m} \binom{m}{k} \alpha^k Ax^{m-k}y^k$, and that

$$Ay^k = 0, k \in \overline{1, m-1} \quad \Rightarrow \quad Ay^{k+s} = 0, s \in \overline{1, m-k} \text{ and } Ay^kx^{m-k} = 0, \forall x \in \mathbb{R}^n.$$

We note that the mixed action on vectors provides alternative versions of angles in the Berwald-Moor relativistic framework ([10], [11]). The following result provides a characterization of recession vectors ([14, Th. 3, p. 1315]):

Theorem 3.1 *Let R be the set of recession vectors of $A \in T^0_m(\mathbb{R}^n)$. Then*

a) $y \in R \Leftrightarrow Ay = 0$, and R is a vector subspace of $\mathbb{R}^n$;
b) any unit vector from R is a $Z-$eigenvector associated to the $Z-$eigenvalue $\lambda = 0 \in \sigma(A)$;
c) any vector of R is orthogonal to all the $Z-$eigenvectors associated to non-zero $Z-$eigenvalues.

As a consequence of the fact that for the Berwald-Moor supersymmetric tensor A given in (1.1), we have $Ay = 0 \Rightarrow y = 0$, a) immediately infers that $R = \{0_{\mathbb{R}^4}\}$.

As well, denoting the base of Σ with $B = R^\perp$, then the *rank of* T is $\dim B$. We note that in the Berwald-Moor case, since $B = \mathbb{R}^4$, we have rank $A = 4$ and hence A is non-singular. Hence Σ is *not* a cylinder. Actually, Σ is a Tzitzeica surface given by $\Sigma: \quad y_1y_2y_3y_4 = c$, which for $c > 0$ has 8 connected sheets.

4 Asymptotic rays

We say that the semi-line $L = \{\alpha y \mid \alpha \geq 0\} \subset \mathbb{R}^m$ with $||y|| = 1$ is an asymptotic ray for $A \in T^0_m(\mathbb{R}^n)$ for Σ if $\alpha y \notin \Sigma, \forall \alpha \geq 0$ and there exists a sequence of points $u^{(k)} \in \Sigma, k \geq 1$, such that $||u^{(k)}|| \to \infty$ and $d(u^{(k)}, L) \to 0$, for $k \to \infty$. We say that an asymptotic ray L of A is of degree d if there exists a sequence of points as above, such that

$$||u^{(k)} - \mathrm{pr}_y u^{(k)}|| = \mathcal{O}\left(||\frac{1}{||u^{(k)}||}u^{(k)} - y||^d\right).$$

It is known that if L is an asymptotic ray for A, then $Ay^m = 0$. From [14], we get

Theorem 4.1 *Let $||y|| = 1$ and $Ay^m = 0$. Then:*

a) L is an asymptotic ray for A iff $y \notin R$;
b) L is an asymptotic ray of degree $1 - \frac{1}{m}$ iff y is not a $Z-$eigenvector associated to the $Z-$eigenvalue $\lambda = 0$;
c) L is an asymptotic ray of degree $1 - \frac{k}{m}$ iff y is a $Z-$eigenvector associated to the $Z-$eigenvalue $\lambda = 0$ and $Ay^{m-k} \neq 0, Ay^{m-k+1} = 0$.

For the Berwald-Moor tensor, one easily finds that the asymptotic rays are characterized by $Ay^4 = y_1y_2y_3y_4 = 0$, and belong to the union of the four coordinate hyperplanes of $\mathbb{R}^4$. Moreover, we have the possible degrees of the asymptotic rays $3/4; 1/2; 1/4$ for $y \in D_0 \backslash D_1$, $y \in D_1 \backslash D_2$ or $y \in D_2 \backslash D_3$, respectively (according to the definitions of degeneracy sets from below.

5 Degeneracy vectors and base index

We have the decomposition $S_0 = R \oplus S_{0B}$, where $S_0 : Ay^m = 0$ and $S_{0B} = S_0 \cap R^\perp$.

Definition. a) A vector $y \in R^\perp$ is called *degeneracy vector of degree k* of A if $Ay^{m-k} = 0$, the set of such vectors being denoted with $D_k, k \in \overline{0, m-2}$.

b) If $D_{k-1} \neq \{0\}$ but $D_k = \{0\}$, then we call $d = k$ *the base index*. If $D_0 = \{0\}$, then we put $d = 0$.

The following result provides a general characterization of degeneracy sets:

Theorem 5.1 *Let $A \in T^0_m(\mathbb{R}^n)$. Then*

a) We have the chain

$$\mathbb{R}^4 \supset D_0 \supset D_1 \supset D_2 \supset \cdots \supset D_{m-2} \supset \{0\};$$

b) $\forall k \in \overline{0, m-2}, (x \in D_k \Rightarrow Span(x) \subset D_k)$;
c) If A is positive or negative semidefinite, then $D_1 \neq \{0\}$;
d) If $k + j \geq m$, then $D_k + D_j \subset D_{k+j-m}$.

For the Berwald-Moor tensor, we have

- $D_0 = \{y | Ay^4 = 0\} = \{y | \exists i \in \overline{1,4}, y_i = 0\}$;
- $D_1 = \{y | Ay^3 = 0\} = \{y | \exists i, j \in \overline{1,4}, i \neq j, y_i = y_j = 0\}$;
- $D_2 = \{y | Ay^2 = 0\} = \{y | \exists i, j, k \in \overline{1,4}, i \neq j \neq k \neq i, y_i = y_j = y_k = 0\}$;
- $D_3 = \{y | Ay = 0\} = \{0\} = R$.

and a) provides the link

$$\mathbb{R}^4 \supset D_0 \supset D_1 \supset D_2 \supset D_3 = \{0\}.$$

Since $D_2 \neq \emptyset$, the base index is $d = 3$. However, item c) does not apply for A, and $D_2 + D_2 \subset D_4$.

We note that the vectors of a degeneracy set D_k provide null Pavlov poly-angles with complimentary $k-$copies of vectors, for $k \in \overline{1,2}$ ([10, 11]).

6 Characterization points

According to [14, Th. 7, p.1324], we have that for $m \geq 2$, $\lambda \in \sigma(A) \cap \mathbb{R}_+^*$ with $x \in \mathbb{R}^n$ associated eigenvector, considering $y = \theta x$ with $\theta = (c/\lambda)^{1/m}$, if $y \in \Sigma$ provides the minimal distance $d(O, \Sigma)$, if except for the value $2-m$, the $(0,2)-$tensor $I - \frac{m-1}{\lambda} Ax^{m-2}$ has all the other eigenvalues positive, and local maximizer for the distance for the similar statement with negative eigenvalues. Further, we have

Theorem 6.1 *Let $A \in T_m^0(\mathbb{R}^n)$, y as above and assume that the minimality condition from above holds. Then, denoting $T_1 = \{u \in \mathbb{R}^n | x \perp u, u = \frac{m-1}{\lambda} Ax^{m-2}u\}$, we have*

a) for $m = 3$, y provides a local minimizer for $d(0, \Sigma)$ iff $Ax^{m-3}u^3, \forall u \in T_1$;

b) for $m \geq 4$, if y provides a local minimizer for $d(0, \Sigma)$, then

$$Ax^{m-3}u^3 = 0, \forall u \in T_1 \text{ and } Ax^{m-4}u^4 \leq 0;$$

c) for $m \geq 4$, if

$$Ax^{m-3}u^3 = 0, \forall u \in T_1 \text{ and } Ax^{m-4}u^4 < 0;$$

then y provides a local minimizer for $d(0, \Sigma)$.

We note that in the Berwald-Moor case, the theorem provides the characterization points previously described in (2.2).

7 Best rank-one approximation

We define the best rank-one super-symmetric approximation of $A \in T_m^0(\mathbb{R}^n)$, as the homogeneous polynomial $\lambda y^m \equiv \lambda y \otimes \cdots \otimes y$ which is global minimizer for the distance $||A - \lambda y^m||_F$ for $\lambda \in \mathbb{R}, ||y||_2 = 1$, where $|| \cdot ||_F$ is the Frobenius norm, and where y^n can be regarded as an $n-$th order $n-$dimensional rank-1 tensor with entries $y_{i_1} \dots y_{i_n}$. A strong result in this respect is ([14, Th. 9, p.1325], [15])

Theorem 7.1 *Consider $A \in T_m^0(\mathbb{R}^n)$. Then:*

a) For $\lambda \in \sigma(A)$ and y its associated $Z-$eigenvector, we have

$$\lambda = Ay^m \text{ and } ||A - \lambda y^m||_F^2 = ||A||_F - \lambda^2 \geq 0.$$

b) the best rank one approximation of A is given by the value $max_{\lambda \in \sigma(A)}$ for y associated to λ.

We note that for the Berwald-Moor tensor, the minimizer $\lambda = 1/16$, is attended at the eigenvectors of λ, by the super-symmetric tensors

$$\hat{A} = \sum_{i_{1,2,3,4} \in \overline{1,4}} \frac{1}{16^2 \cdot 4!} \varepsilon_{i_1}\varepsilon_{i_2}\varepsilon_{i_3}\varepsilon_{i_4} dy_{i_1} \otimes dy_{i_2} \otimes dy_{i_3} \otimes dy_{i_4},$$

where $\varepsilon_{1,2,3,4} \in \{\pm 1\}, \varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4 = 1$, attached to the quartic forms

$$\hat{A}y^4 = \frac{1}{16^2}(\varepsilon_1 y_1 + \varepsilon_2 y_2 + \varepsilon_3 y_3 + \varepsilon_4 y_4)^4.$$

One can show that the minimization problem defined above is equivalent to the dual problem of maximizing

$$f(y) = \sum_{i_1,\dots,m = \overline{1,n}} A_{i_1 \dots i_m} y_{i_1} \dots y_{i_m} = \langle A, y^{*m} \rangle, \quad \text{for } ||y||_2 = 1,$$

which turns out to be equivalent to the maximization of the Rayleigh quotient.

The best rank-one approximation has several notable applications in signal processing (e.g., [7,8]).

Conclusions

The Berwald-Moor supersymmetric tensors is discussed in terms of multivariate homogeneous polynomials. We find the eigenvalues and eigenvectors, the recession and degeneracy vectors, characterization points, rank, asymptotic rays, base index and the best rank-one approximation. As well, several relations to the Berwald-Moor poly-angles and real-world applications are included.

Acknowledgement. The present work was developed as part of the Romanian Academy Grant No. 5 / 5.02.2008.

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