Part I. Pseudo-Finsler Geometry

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1 Finsler structures. Finsler metric tensor field

Definitions. a) We call *Finsler structure* a couple (M, F), where M is a real $n-$dimensional C^∞ differentiable manifold and $F : TM \to \mathbb{R}_+$ is a mapping (called *Finsler fundamental function* or *Finsler norm*), which obeys the following properties:

1. F is C^∞ on $\widetilde{TM} = TM \backslash \{0\} \equiv \bigcup\limits_{x \in M} \{(x, y) \mid y \in T_x M, y \neq 0_x\}$ (F is smooth on the tangent space without the image of the null section);

2. $F(x, \lambda y) = \lambda F(x, y)$, $\forall \lambda \in [0, \infty)$ (F is positive homogeneous of degree one in y, i.e., on the fibres of the tangent bundle (TM, π, M));

3. for all $(x, y) \in TM$, the functions
$$g_{ij} = \frac{1}{2} \frac{\partial^2 F^2}{\partial y^i \partial y^j} \tag{1}$$
provide a symmetric positive definite matrix. They are the components of the *metric Finsler tensor field*
$$g = g_{ij}(x, y)\; dx^i \otimes dx^j.$$

b) We call *the indicatrix* of the Finsler structure (M, F) at the point $x \in M$, the set of unit tangent vectors $I_x = \{y \in T_x M \mid F(x, y) = 1\}$.

Remarks. a) If 3) is replaced with the (weaker) condition that g should be non-degenerate and of constant signature (not necessarily positive-definite and hence F not necessarily non-negative), then the structure (M, F) is called *pseudo-Finsler structure.*

b) If 2) is supplemented with the *reversibility condition*, i.e., $F(x, y) = F(x, -y), \forall (x, y) \in TM$, then F becomes homogeneous of first order, i.e., $F(x, \lambda y) = \lambda F(x, y)$, $\forall \lambda \in \mathbb{R}$. This is the case of (e.g.) the Riemannian metric $F = \alpha$.

c) If F is no longer non-negative, or not defined on the whole space TM (being defined only on certain distributions of the tangent bundle), then there appear examples of new (generalized) Finsler functions, e.g., the Kropina metric $F = \alpha^2/\beta$, or the Shimada pseudo-Finsler metric $F(x, y) = \sqrt[m]{a_{i_1 \dots i_m}(x) y^{i_1} \cdot \ldots \cdot y^{i_m}}$, having as particular case the Berwald-Moor metric $F(y) = \sqrt[n]{y^1 \cdot \ldots \cdot y^n}$.

2 The non-linear connection of a Finsler structure

Consider the tangent bundle (TM, π, M). Then a basic tool which allows to introduce a structure of vector bundle on $(TTM, d\pi, TM)$ with structural group $GL_n(\mathbb{R}) \times GL_n(\mathbb{R})$ is *the non-linear connection* $N = \{N^a_i\}$, which provides a *local* adapted basis for the module of sections of the subbundles $(HTM, d\pi, TM)$ and $(VTM, d\pi, TM)$ of the Whitney splitting $(TTM = HTM \oplus VTM, d\pi, TM)$, namely
$$\left\{\delta_i = \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^a_i(x, y) \frac{\delta}{\delta y^i}\right\} \subset \Gamma(HTM), \ \left\{\dot{\partial}_i = \frac{\partial}{\partial y^a}\right\} \subset \Gamma(VTM), \tag{2}$$

where $VTM = \text{Ker } d\pi$. There exists a canonical nonlinear connection, provided by the Finsler fundamental function F, given by (Kern's formula):

$$N_i^a = \frac{\partial G^a}{\partial y^i}, \tag{3}$$

where g^{ij} are the components of the matrix dual (inverse) to g_{ij}, i.e., $g^{is}g_{sj} = \delta^i_j$ and

$$G^a = \frac{1}{2} g^{as} \left(\frac{\partial^2 L}{\partial y^s \partial x^j} y^j + \frac{\partial L}{\partial x^s} \right). \tag{4}$$

3 Linear connections and covariant derivation laws

The components $\nabla = \{\tilde{F}^i_{jk}; \tilde{C}^i_{jk}\}$ of a linear connection which preserves the $h-$ and $v-$distributions (called *linear d-connection*), locally described by (2), computed with respect to the local adapted basis of fields determined by a fixed non-linear connection N

$$\{\delta_1, \ldots, \delta_n; \dot{\partial}_1, \ldots, \dot{\partial}_n\}$$

are described by the relations

$$\nabla_{\delta_k} \delta_j = \tilde{F}^i_{jk} \delta_i; \;\; \nabla_{\dot{\partial}_c} \dot{\partial}_b = \tilde{C}^a_{bc} \dot{\partial}_a. \tag{5}$$

These components split into the $h-$components $\{F^i_{jk}\}$ and the $v-$components $\{C^a_{bc}\}$.

The non-trivial components of the torsion $T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$ of ∇ are Finsler tensor fields, provided by the formulas

$$\begin{aligned} T(\delta_k, \delta_j) &= T^i_{jk} \delta_k + R^a_{jk} \dot{\partial}_a \\ T(\delta_k, \dot{\partial}_b) &= \tilde{C}^a_{bk} \delta_a + P^a_{bk} \dot{\partial}_a \\ T(\dot{\partial}_c, \dot{\partial}_b) &= S^a_{bc} \dot{\partial}_a, \end{aligned} \tag{6}$$

These non-trivial torsion fields $T \equiv \{T^i_{jk},\, R^a_{jk},\, P^a_{bk},\, \tilde{C}^i_{ja},\, S^a_{bc}\}$ described in (6), have the explicit expressions in terms of the connection coefficients $\nabla = \{\tilde{F}^i_{jk}; \tilde{C}^i_{jk}\}$ given by (5), as follows

$$T^i_{jk} = \tilde{F}^i_{jk} - \tilde{F}^i_{kj}, \;\; R^a_{jk} = \delta_k N^a_j - \delta_j N^a_k, \;\; P^a_{bk} = \tilde{F}^a_{bk} - \dot{\partial}_b N^a_k, \;\; \tilde{C}^i_{ja}, \;\; S^a_{bc} = \tilde{C}^a_{bc} - \tilde{C}^a_{cb}. \tag{7}$$

We note that *the Cartan tensor field* which is defined in (11) is totally symmetric in all its indices and that (M, F) is a Riemannian structure (i.e., g_{ij} depend on x only) iff $C_{ijk} \equiv 0$.

As well, for a Finsler linear d-connection ∇, one can define the associated $h-$ and the $v-$covariant derivatives, as follows

$$\left\{ \begin{aligned} T^{i_1 \ldots i_p}_{j_1 \ldots j_q \,|\, k} &= \frac{\delta T^{i_1 \ldots i_p}_{j_1 \ldots j_q}}{\delta x^k} + \sum_{s=\overline{1,p}} T^{i_1 \ldots t_s \ldots i_p}_{j_1 \ldots j_q} \tilde{F}^{i_s}_{t_s k} - \sum_{s=\overline{1,q}} T^{i_1 \ldots i_p}_{j_1 \ldots t_s \ldots j_q} \tilde{F}^{t_s}_{j_s k} \\ T^{i_1 \ldots i_p}_{j_1 \ldots j_q} \Big|_k &= \frac{\partial T^{i_1 \ldots i_p}_{j_1 \ldots j_q}}{\partial y^k} + \sum_{s=\overline{1,p}} T^{i_1 \ldots t_s \ldots i_p}_{j_1 \ldots j_q} \tilde{C}^{i_s}_{t_s k} - \sum_{s=\overline{1,q}} T^{i_1 \ldots i_p}_{j_1 \ldots t_s \ldots j_q} \tilde{C}^{t_s}_{j_s k}, \end{aligned} \right.$$

for any Finsler tensor field of type (p,q). The connection ∇ is said to be $h-$*metrical* iff $g_{ij|k} = 0$ and $v-$*metrical* iff $g_{ij}|_k = 0$. There exist several well-known linear connections in Finsler Geometry, used in applications. None of these connections has the two basic qualities of the Levi-Civita connection from Riemannian Geometry (metricity and symmetry). The reasons for which such a connection ceases to exist on the tangent space TM, is that the natural extra requirement - that of preserving the $h-$ and $v-$distributions locally described by (2), is fulfilled. Below is a table which contains four linear connections which obey this extra property, but fail to be completely metric and symmetric.

Linear connection	Coefficients	m	m	s	s	s	s	s
	$\nabla = \{\tilde{F}^i_{jk}, \tilde{C}^i_{jk}\}$	$h-$	$v-$	$hh-h$	$hh-v$	$hv-h$	$hv-v$	$vv-v$
Cartan	$\overset{C}{\nabla} = \{F^i_{jk}, C^i_{jk}\}$	Y	Y	Y	N	N	N	Y
Chern-Rund	$\overset{CR}{\nabla} = \{F^i_{jk}, 0\}$	Y	N	Y	N	Y	N	Y
Berwald	$\overset{B}{\nabla} = \{G^i_{jk}, 0\}$	N	N	Y	N	Y	Y	Y
Hashiguchi	$\overset{H}{\nabla} = \{G^i_{jk}, C^i_{jk}\}$	N	Y	Y	N	N	Y	Y

The basic formulas which provide the components of the four connections from above are:

$$G^i_{jk} = \dot{\partial}_j N^i_k \tag{8}$$

$$F^i_{jk} = \frac{1}{2} g^{is}(\delta_j g_{sk} + \delta_k g_{sj} - \delta_s g_{jk}) \tag{9}$$

$$C^i_{jk} = \frac{1}{2} g^{is}(\dot{\partial}_j g_{sk} + \dot{\partial}_k g_{sj} - \dot{\partial}_s g_{jk}) = g^{is} C_{isk}, \tag{10}$$

where

$$C_{isk} = \frac{1}{2}\frac{\partial g_{is}}{\partial y^k} = \frac{1}{4}\frac{\partial^3 F^2}{\partial y^i \partial y^s \partial y^k}. \tag{11}$$

4 Invariants of a Finsler structure

The invariants of Riemann type of a Finsler structure (M, F) are the following

$$\begin{cases} g = g_{ij} dx_i \otimes dx_j, & g_{ij} = \dfrac{1}{2}\dfrac{\partial^2 F^2}{\partial y^i \partial y^j} \\ \eta = \eta_i dx^i, & \eta_i = \partial F / \partial y^i. \end{cases}$$

We note that these are 0-homogeneous Finsler tensor fields of type (0,2), (0,3) and (0,1), respectively. The invariants of non-Riemannian type are

$$\begin{cases} A_{ijk} dx^i \otimes dx^j \otimes dx^k, & A_{ijk} = \frac{F}{4}\dfrac{\partial^3 F^2}{\partial y^i \partial y^j \partial y^k} = \dfrac{F}{2}\dfrac{\partial g_{ij}}{\partial y^k} \\ A_i dx^i, & A_i = g^{jk} A_{ijk} \\ \tau = \dfrac{\sqrt{\det(g_{ij})_{i,j=\overline{1,n}}}}{\sigma(x)}, & \sigma(x) = \dfrac{Vol(S^n(1))}{Vol(\{y \in \mathbb{R}^n \mid F(x, y^i \frac{\partial}{\partial x^i}) \le 1\})}. \end{cases}$$

We note that these are 0-homogeneous Finsler tensor fields of type (0,1) and (0,0), respectively.

5 Examples of Finsler and pseudo-Finsler structures

I. The Finsler metric function $F_R(x,y) = \sqrt{g_{ij}(x) y^i y^j}$, where $\{g_{ij}(x)\}$ are the coefficients of a Riemannian metric on M, practically represents the Banach norm associated to the Riemannian structure $g = g_{ij}(x) dx^i \otimes dx^j$ of (M, g). In particular, for $g_{ij} = \delta_{ij}$, one gets for the Finsler metric $F(y) = \sqrt{(y^1)^2 + \ldots + (y^n)^2}$ of the Euclidean structure (M, δ_{ij}).

Similarly, the pseudo-Finsler metric function $F_R(x,y) = \sqrt{g_{ij}(x) y^i y^j}$, where $\{g_{ij}(x)\}$ are the coefficients of a pseudo-Riemannian metric, is the pseudo-Riemannian norm attached to the pseudo-Riemannian structure $g = g_{ij}(x) dx^i \otimes dx^j$ of (M, g). In particular, for $g_{ij} = \varepsilon_i \delta_{ij}$, where $\varepsilon_1 = -1, \varepsilon_2 = \ldots = \varepsilon_n = 1$, one gets for the Minkowski pseudo-Finslerian norm $F(y) = \sqrt{-(y^1)^2 + (y^2)^2 + \ldots + (y^n)^2}$ attached to the pseudo-Euclidean Minkowski structure $(M, \varepsilon_i \delta_{ij})$.

II. The Randers metric $F = \alpha + \beta$, with

$$\alpha = \sqrt{a_{ij}(x)y^i y^j}, \quad \beta = b_k(x)y^k, \tag{12}$$

where $a = a_{ij}(x)dx^i \otimes dx^j$ is a Riemannian metric on M and $b_k(x)dx^k$ is a 1-form on M, has the property that for $\beta = 0$, i.e., for $b_k = 0, \forall k \in \overline{1,n}$ reduces to the Finsler Riemannian structure $(M, a \equiv \{a_{ij}\})$.

The Randers spaces were first introduced by Randers in 1941, and further studied by Roman Ingarden (who gave their actual name in 1957), Ralf G. Beil, Aurel Bejancu, Ioan Buc'ataru, Vasile Sabău, etc. The Randers spaces have been shown to provide an alternative model for unified gravitation and electromagnetism.

Homework. Denote $||b||_a = ||b_k(x)dx^k||_a = a^{ij}(x)b_i(x)b_j(x)$, and check that for $||b||_a < 1$ one gets a Finsler structure with compact indicatrix $I_x, \forall x \in M$ which encloses a convex body, while for $||b||_a > 1$, F defines a pseudo-Finsler structure with non-bounded indicatrix.

III. Particular cases of Randers metrics:

a) The *generalized Funk metric* given by

$$F(x,y) = \frac{\sqrt{(||x||^2\langle a,y\rangle - 2\langle a,x\rangle\langle x,y\rangle)^2 + ||y||^2(1-||a||^2||x||^4)}}{1-||a||^2||x||^4} - \frac{||x||^2\langle a,y\rangle - 2\langle a,y\rangle\langle x,y\rangle}{1-||a||^2||x||^4},$$

where $x \in D = B(0, ||a||^{-1/2}) \subset \mathbb{R}^n, y \in T_xD, a \in \mathbb{R}^n$.

Homework. Show that (D,F) is a Randers space of scalar curvature

$$K = 3||F||^{-1}\langle a,y\rangle + 3\langle a,x\rangle^2 - 2||a||^2||x||^2,$$

and that it has isotropic $S-$curvature, given by $S = (n+1)\underbrace{\langle a,x\rangle}_{c(x)}F$.

b) The Finsler space $(M = B^n(1/\sqrt{-\mu}) \subset \mathbb{R}^n, F)$ with $\mu < 0$ and

$$F_\mu(x,y) = \frac{(||y||^2 + \mu(||x||^2||y||^2 - \langle x,y\rangle^2))^{1/2}}{1+\mu||x||^2}, \ \forall y \in T_xM.$$

Homework. Show that (D,F) is a Randers space. Compute its flag and $S-$curvatures.

c) The Finsler space $(M = \mathbb{R}^2, F)$ with $\lambda \in [0,\infty), k \geq 1$ and

$$F_{\lambda,k}(y) = \sqrt{a^2 + b^2 + \lambda(a^{2k} + b^{2k})^{1/k}}, \ \forall y = (a,b) \in T_{(x_1,x_2)}\mathbb{R}^2, (x_1,x_2) \in \mathbb{R}^2.$$

Homework. Show that (D,F) is a Randers space. Compute its flag and $S-$curvatures. Show that for $\lambda = 0$, the Finsler fundamental function provides the Euclidean norm on $\mathbb{R}^2$.

IV. The Kropina pseudo-Finsler metric $F = \alpha^2/\beta$, has been used in irreversible thermodynamics, as

$$F(y) = y_{n+1}^{-1}\sum_{k=1}^{n+1} y_i^2, \ y = (y_1, \ldots, y_{n+1}) \in D \subset \mathbb{R}^{n+1},$$

and in Fisher statistic Finsler geometry.

V. The (α,β)-Finsler metrics are the ones with the fundamental function of the form

$$F(x,y) = \alpha\Phi\left(\frac{\beta}{\alpha}\right),$$

with α, β given by (12), where Φ is a positive smooth function. We note that for $\Phi(s) = 1$, $\Phi(s) = 1+s$ and $\Phi(s) = s^{-1}$, we accordingly obtain the Riemannian spaces $F = \alpha$, the Randers spaces $F = \alpha + \beta$, and the Kropina spaces $F = \alpha^2/\beta$.

6 Geodesics in (pseudo-)Finsler spaces

A curve $x(t)$ is called *geodesic* of the Finsler space (M,F) iff it satisfies the equations of geodesics

$$\ddot{x}^i + \gamma^i_{jk}(x,\dot{x})\dot{x}^j\dot{x}^k = 0, \quad i = \overline{1,n}, \tag{13}$$

where $\ddot{x} = \dfrac{d^2x}{dt^2}, \dot{x} = \dfrac{dx}{dt}$ and $\gamma^i_{jk} = |^{\,i}_{jk}|$ are *the generalized Christoffel symbols of the second kind* provided by the Christoffel symbols of the first kind $|jk;s|$,

$$|^{\,i}_{jk}| = g^{is}|jk;s|, \quad |jk;s| = \frac{1}{2}(\partial_j g_{sk} + \partial_k g_{sj} - \partial_s g_{jk}), \quad i,j,k = \overline{1,n}.$$

Remark. Like in the (sub-)Riemannian case, when the tensor g_{ij} is *degenerate*, then the geodesics still can be defined by avoiding the usage of the Christoffel symbols of the II-nd kind (which involve the existence of the inverse tensor g^{ij}), by using the lowered index version of (13) and the symbols of I-st kind, as

$$g_{is}\ddot{x}^s + |jk;i|\dot{x}^j\dot{x}^k = 0, \quad i = \overline{1,n}. \tag{14}$$

We note that γ^i_{jk} depend on (x,y), and that this provides a canonic non-linear connection provided by F (called *the Cartan non-linear connection*), whose coefficients are given by:

$$N^a_i = \frac{\partial(\gamma^a_{jk}y^jy^k)}{\partial y^i},$$

that the coefficients $G^i = \frac{1}{2}\gamma^i_{jk}y^iy^k$ (called *spray coefficients*) are 2-homogeneous in y, and that the equations (13) rewrite as

$$\ddot{x}^i + 2G^i(x,\dot{x}) = 0, \quad i = \overline{1,n}.$$

Homework. Using the MAPLE software, compute the generalized Christoffel symbols, the equations of geodesics and plot (as concurrent sheaf and as transversal net) the geodesics of the Euclidean space $\mathbb{R}^2$, of the parametrized sphere $\Sigma = Im\ r,\ r(u,v) = (\cos u \sin v, \sin u \sin v, \cos v),\ (u,v) \in [0,2\pi] \times [0,\pi]$, and for the Finsler structures from examples III (choose $n = 2$ and convenient initial data for solving/plotting the corresponding 2-nd order SODE).

7 The flag curvature of a Finsler space

For a given Finsler linear d-connection $\nabla = \{\tilde{F}^i_{jk}; \tilde{C}^i_{jk}\}$, one can easily compute the components of the curvature $R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$, which, due to the property that ∇ preserves the $h-$ and the $v-$distributions, provides three nontrivial sets of coefficients, via:

$$R(\delta_l,\delta_k)\delta_j = R^i_{jkl}\delta_i, \quad R(\dot{\partial}_c,\delta_k) = P^a_{bkc}\dot{\partial}_a, \quad R(\dot{\partial}_d,\dot{\partial}_c) = S^a_{bcd}\dot{\partial}_a.$$

Homework. Check that the non-linear connection $N = \{N^a_i\}$ given by Kern's relations (3)-(4) satisfies $N^a_i = \frac{\partial(\gamma^a_{bc}y^by^c)}{\partial y^i}$ and that its curvature R^a_jk provided by (7) satisfies $\frac{\partial R^a_{jk}}{\partial y^l} = R^a_{jkl}$.

By means of this curvature tensor, one can easily define the $h-$curvature of Riemann type, $K(X,Y,Z,V) = g(R(Z,V)Y,X)$, and further, for a family of oriented 2-tangent planes generated by two independent vector fields X,Y, the associated sectional curvature

$$K_{(x,y)}(X,Y) = \frac{K(X,Y;X,Y)}{Gram_{det}(X,Y)}, \quad \text{where } Gram_{det}(X,Y) = \begin{vmatrix} g(X,X) & g(X,Y) \\ g(Y,X) & g(Y,Y) \end{vmatrix}.$$

We note that this sectional curvature generally depends on the support element $(x,y) \in TM$, hence there exist *three tangent vectors* $y, X_x, Y_x \in T_xM$ which take part in this definition, in the proper

Finslerian case. Still, in the particular Riemannian case, for $x \in M$ fixed, this depends just on the vectors $X_x, Y_x \in T_xM$ and provide the classical sectional curvature of the subjacent integral local submanifold $\Sigma_{X,Y} \subset M$, whose tangent plane is spanned by these two vectors, and the formula provides by the Egregium theorem *the Gauss curvature* of $\Sigma_{X,Y}$.

In the proper Finsler case we reduce the number of vectors from three to two, by replacing $X \to y, Y \to X$, and we get the so-called *flag curvature* associated to the 2-plane spanned by $\{y, X\}$,

$$K_{(x,y)}(X) = \frac{K(y, X; y, X)}{Gram_{det}(y, X)}.$$

The Finsler space is called *of scalar (flag) curvature* if $K_{(x,y)}(X)$ does not depend on the field X, and *of constant flag curvature* if $K_{(x,y)}(X)$ is independent on both y and X.

The role played by the flag curvature is significant in Finsler geometry. Among the latest related results, we note the following:

Theorem (Rademacher 2004). Let (M, F) be a simply connected compact Finsler structure such that $\dim M \geq 3$. If the flag curvature satisfies the condition $(1 - \frac{1}{1+\lambda}) < K_{(x,y)}(X) \leq 1$, $\forall x \in M, \forall y \in T_xM, \forall X_x \in T_xM$, where $\lambda = \sup\limits_{F(x,y)=1} \dfrac{F(x,-y)}{F(x,y)} = \sup\limits_{y \in I_x} \dfrac{F(x,-y)}{F(x,y)}$, then M is homeomorphic to the sphere S^n.

Theorem (Z. Shen 2005). Let (M, F) be a compact Finsler structure such that $\dim M \geq 3$, of scalar negative curvature. Then F is a Randers metric.

8 The Legendre transform of a Finsler structure

Definition. Let (M, F) be a Finsler structure. Then the mapping $\mathcal{L} : TM \to T^*M$ locally defined by

$$T(x, y) = (x, p), \text{ where } p_i = g_{ij}(x, y)y^j, \ \forall (x, y) \in TM,$$

is called *the Legendre transform* of the given Finsler structure.

It can be proved that L is a local diffeomorphism, and that it transforms the Finsler norm F into *the dual norm* $F^* : T^*M \to \mathbb{R}_+$ defined by $F^*(x, p) = \sup\limits_{F(x,y)=1} |p(y)| = \sup\limits_{y \in I_x} |p(y)|$.

Remarks. a) The *dual indicatrix*, i.e., the indicatrix $I_x^* = \{p \in T^*M \mid F^*(x, p) = 1\}$ of F^* can be locally described by its Cartesian implicit equation which is obtained by eliminating the parameters $y^1, \ldots, y^n$ from the system of $n + 1$ relations

$$\begin{cases} F(x; y^1, \ldots, y^n) = 1 \\ F_{y^1}(x; y^1, \ldots, y^n) = 0 \\ \ldots \\ F_{y^n}(x; y^1, \ldots, y^n) = 0 \end{cases}.$$

b) One can easily see that the (local) inverse $\mathcal{L}^{-1} : T^*M \to TM$ is generally hard to obtain. Namely, the pre-image $(x, y) \in TM$ of $(x, p) \in T^*M$ can be determined by finding the unique solution $y = (y^1, \ldots, y^n)$ of the system of highly non-linear equations

$$\begin{cases} g(x, y)_{11} \cdot y^1 + \ldots + g(x, y)_{1n} \cdot y^n = p_1 \\ \ldots \\ g(x, y)_{n1} \cdot y^1 + \ldots + g(x, y)_{nn} \cdot y^n = p_n. \end{cases}$$

c) In the (particular) Riemannian case, when $g(x,y)_{ij}$ depends only on $x \in M$, this system is linear and compatible, with the unique solution provided by $y^i = g^{ij}p_j, i = \overline{1,n}$. We then easily note that the Legendre transform becomes in this case a vector bundle isomorphism, acting on the fibres as

$$\mathcal{L}_x : T_xM \to T_x^*M,\ \mathcal{L}_x(y) = p, \quad p_i = g_{ij}y^j,$$

thus "lowering" the index ($L_x = L_\flat$). As well, its inverse acts on fibres as

$$\mathcal{L}_x^{-1} : T_x^*M \to T_xM,\ \mathcal{L}_x^{-1}(p) = y, \quad y^i = g^{ij}p_j,$$

thus "raising" the index ($L_x^{-1} = L_\#^{-1}$). Hence, often L is called (in the Riemannian case), *the musical isomorphism.*

Homework. Show that the Cartesian indicatrix I_x and the dual indicatrix I_x^* ($x \in \mathbb{R}^4$) of the Berwald-Moor pseudo-Finsler metric are Tzitzeica surfaces, provided by the same Cartesian equations: $I_x : y^1y^2y^3y^4 = 1$; $I_x^* : p_1p_2p_3p_4 = 1$.

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Exam quizzes

I. Solve several selected by the examiner homeworks.

II. Thoroughly comment the MAPLE code related to geodesic plots.

III. Provide evidence of assimilating basic knowledge regarding one selected by the examiner section.

IV. Check that the provided examples of Finsler structures satisfy the corresponding axioms for (pseudo-)Finsler structures.

Part II. Symmetries and transformation groups

Lecture notes - Balan Vladimir * Autumn School - Fryazino

December 18, 2008

1 Lie algebras and Lie groups

Lie Algebras

Definition

Lie algebra: a vector space V with $[\,\cdot\,,\,\cdot\,] : V \times V \to V$ satisfying:
a) $\mathbb{R}-$bilinearity
b) $[a, b] = -[b, a]$ (skew symmetry)
c) $[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0$ (the Jacobi identity).

Remark

Using $ad : V \to End(V)$, $\;\text{ad } a(b) = j_{a*}(b) = [a, b]$ $(j_a(b) = aba^{-1})$, the Jacobi identity confirms that ad a is a *derivation* $\forall a \in V$, i.e.

$$\text{ad } a([b, c]) = [\text{ad } a(b), c] + [b, \text{ad } a(c)]$$

Examples of Lie algebras

- $\mathbb{R}^3$ with $[u, v] = u \times v$ (the cross product)
- $End(V)$ with $[A, B] = A \circ B - B \circ A$ (using composition)
- $M_n(K)$ with $[A, B] = A \cdot B - B \cdot A$ (using matrix multiplication)
- $\mathcal{X}(M)$ with $[X, Y] = (X^i \cdot \partial_i(Y^j) - Y^i \cdot \partial_i(X^j))\partial_i, \forall X, Y \in \mathcal{X}(U)$
- for $M = G$ Lie group, $(\mathfrak{g}, [\,\cdot\,,\,\cdot\,])$, $\mathfrak{g} \equiv T_eG \equiv \mathcal{X}_r(G) \equiv \mathcal{X}_l(G)$
- induced Lie bracket on submanifolds of M

Lie groups

Definition 1. *a) Lie group $(G,\,\cdot\,)$ group, diff. manifold with group operation and inversion both smooth mappings*

b) Lie subgroup - a subset $H \subset G$ which is both algebraic subgroup and submanifold of G

Remark

Any Lie group is an orientable and parallelizable manifold

Examples of Lie groups

- $(K^n, +)$ and $(K^*,\,\cdot\,)$, $K \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$
- $(M_n(V), +)$; $(GL(V),\, \circ\,)$
- $(GL(n, K) = GL(K^n),\,\cdot\,)$, $K \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$
- $\text{Aff}\,(S) = GL(V) \circledS T$, ($V =$ the linear space associated to the affine space S)

Examples

- G finite or countable (with the discrete topology of 0-dimensional manifolds)
- $G, G\,\mathrm{pr} \;\Rightarrow\; G \times G\,\mathrm{pr}$; consequence: $K^m \times T^n$ where $T^n = \underbrace{S^1 \times \cdots \times S^1}_{\text{n times}}$
- any closed subgroup of a Lie group
- H normal subgroup of G ($H \triangleleft G \Leftrightarrow \forall g \in G, gHg^{-1} = H$)

Examples

- real Lie groups: subgroups of $GL(n, \mathbb{R})$: $O(n), SL(n, \mathbb{R}), SO(n) = O(n) \cap SL(n, \mathbb{R})$
- complex Lie groups: subgroups of $GL(n, \mathbb{C})$: $U(n), SL(n, \mathbb{C}), SU(n) = U(n) \cap SL(n, \mathbb{C})$
- $T = T^1 = \{z \in \mathbb{C}^* \mid |z| = 1\} = S^1 \subset \mathbb{C}^*$
- W additive subspace of V; $(K^*)^n \subset GL(K^n)$
- the Heisenberg-Weil triangular subgroups of $GL(n, K)$ (used in second quantization and coherent states):

$$H_R = \{(a_{ij}) \mid a_{ii} = 1, a_{ij} = 0, \forall i > j\},$$
$$H_L = \{(a_{ij}) \mid a_{ii} = 1, a_{ij} = 0, \forall i < j\}$$

1.1 Homomorphisms of Lie groups

Definition. $G \xrightarrow{\varphi} G\,\mathrm{pr}$ homomorphism of Lie groups if φ is smooth and algebraic morphism.

Examples.

- $x \in \mathbb{R} \to e^x \in \mathbb{R}_+^*$
- $A \in GL(n, K) \to \det A \in K^*$
- $G \xrightarrow{\varphi} G$, $\varphi(g) \in \{ag, ga, aga^{-1}\}$ for some $a \in G$
- $x \in \mathbb{R} \to e^{ix} \in S^1 \in \mathbb{C}^*$
- $(A, b) \in \mathrm{Aff}\,(S) \to A \in GL(V)$ (V associated to the affine space S)
- any algebraic morphism from G finite or countable to a Lie group $G\,\mathrm{pr}$

2 Lie algebras of Lie groups

- $sl(n, K) = \{A \in M_n(K) |\; Tr(A) = 0\}$, $K \in \{\mathbb{R}, \mathbb{C}\}$
- $o(n, K) = \{A \in M_n(K) | A^t = -A\}$, ($K \in \{\mathbb{R}, \mathbb{C}\}$, $o(n) = o(n, \mathbb{R})$)
- $so(n, K) = o(n, K) \cap sl(n, K)$
- $o(p, q) = \{A \in M_n(\mathbb{R}) | AJ + JA^t = 0\}$
  $= \{AJ | A^t = -A\} = o(n)J, \quad J = \mathrm{diag}\,(I_p, -I_q)$.
- $so(p, q) = o(p, q) \cap sl(n, \mathbb{R})$

Examples of Lie algebras

- $u(n) = \{A \in GL(n, \mathbb{C}) |\ \bar{A}^t = -A\}$ (skew-Hermitian)
- $su(n) = u(n) \cap sl(n, \mathbb{C})$
- $u(p,q) = \{A \in M_n(\mathbb{C}) | AJ + J\bar{A}^t = 0\}$
  $= \{AJ | \bar{A}^t = -A\} = su(n)J, \quad J = \text{diag}\ (I_p, -I_q).$
- $su(p,q) = u(p,q) \cap sl(n, \mathbb{C})$
- Heisenberg-Weil groups have the Lie algebras $Lie(H_U) = \{(a_{ij})_{i,j=\overline{1,n}} \mid a_{ij} = 0, \forall i \geq j\}$, similar $Lie(H_L)$.

Examples of Lie algebras

Particular cases

- $so(3, \mathbb{R}) = \text{Span}\,(\{[R_{x,\pi/2}], [R_{y,\pi/2}], [R_{z,\pi/2}]\}),\ [\,\cdot\,,\,\cdot\,]$
- $\mathbb{R}^3 = \text{Span}\,(\{e_1, e_2, e_3\}),\ \times$
- $su(2) = \text{Span}\ \left(\left\{\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\right\}\right),\ [\,\cdot\,,\,\cdot\,]$
- The three Lie algebras are isomorphic.
- $sl(2, \mathbb{R}) = \text{Span}\ \left(\left\{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right\}\right)$

3 Lie transformation groups

Transformation groups in plane

- $T \equiv \mathbb{R}^2$ - translations $x \to x + x_0$
- $H \equiv \mathbb{R}^*_+$ - homotheties $x \to kx$
- $H \circledS T$ semidirect product ($(a, x) \circ (b, y) = (ab, x + ky)$)

- $GL_2(\mathbb{R})$ - linear isomorphisms of $\mathbb{R}^2$
- $\text{Aff}\,(\mathbb{R}^2) = GL_2(\mathbb{R}) \circledS T$ - the affine group

Isometries

- $O(2) = \{A \in GL_2(\mathbb{R}) \mid A^t A = I_2\}$ - orthogonal transformations (linear isometries)
- $\text{Iso}\,(\mathbb{R}^2) = O(2) \circledS T$ - isometries
- the special orthogonal group (the connected component of $O(2)$, positive linear isometries)

$$\begin{aligned} SO(2) &= \{A \in O(2) \mid \det(A) = 1\} \\ &= \left\{[R_\theta] = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \mid \theta \in \mathbb{R}\right\}, \end{aligned}$$

- $SO(2) \circledS T$ (proper movements, positive isometries)

Conformal transformations

- Conf $(\mathbb{R}^2) = (H \times O(2))_{\circledS} T$ - the conformal group
- Conf $_+(\mathbb{R}^2) = (H \times SO(2))_{\circledS} T$ - orientation preserving (positive) conformal transformations
- $O(2) = SO(2) \cup \{A \cdot \text{diag}\,(1, -1) \mid A \in SO(2)\}$
- glide reflections (negative isometries; not a group !):

$$(O(2) \backslash SO(2))_{\circledS} T = \{symm_\Delta \circ T_{v||v_\Delta} \mid \Delta \subset \mathbb{R}^2\}$$

- Conf $_+(\mathbb{R}^2) \ \sim \ \text{Aff}\,(1, \mathbb{C}) = \{z \to wz + w_0 \mid z \in \mathbb{C}^*, w_0 \in \mathbb{C}\}$
- Conf $(\mathbb{R}^2) \ \sim \ \{z \to wz + w_0 \ \textbf{or} \ z \to w\bar{z} + w_0 \mid z \in \mathbb{C}^*, w_0 \in \mathbb{C}\}$

Projective and concircular transformations

- $PGL(2, \mathbb{R}) = \{x \to \pi_{12}([A(x^t, 1)]) \mid A \in GL(3, \mathbb{R})\}$ (the projective transformations of the plane)
- the circular transformations:

$$Circ_2 = PGL(1, \mathbb{C}) \cup \underbrace{\{z \to \varphi(\bar{z}) \mid \varphi \in PGL(1, \mathbb{C})\}}_{PGL(1,\mathbb{C})\,\mathrm{pr}}$$

- $PGL(1, \mathbb{C})$ = proper fractional linear transformations = inversions $\circ$ spiral similarities $\circ$ reflections $\circ$ translations
- $PGL(1, \mathbb{C})\,\mathrm{pr}$ = improper fractional linear transformations = inversions $\circ$ spiral similarities $\circ$ translations
- $SO(2)_{\circledS} H$ - spiral similarities of center O
- $x \to \rho \cdot x / ((x^1)^2 + (x^2)^2)$, inversion of center O

Addenda - other Lie group transformations

Isometries of $\mathbb{R}^n$

- $n = 3$: positive isometries (proper motions) $\subset$ isometries $\subset$ affine transformations

$$SO(3)_{\circledS} T \subset O(3)_{\circledS} T \subset GL_3(\mathbb{R})_{\circledS} T$$

- isometries of $\mathbb{R}^n$:

$$SO(n)_{\circledS} T \subset O(n)_{\circledS} T \subset GL(n, \mathbb{R})_{\circledS} T$$

- Galilean transformations of classical mechanics (in $\mathbb{R}^4_{3,1}$)

$$(t, x) \to (t, Ax + x_0 - vt) \quad \sim \quad \begin{pmatrix} A & -v & x_0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in PGL(\mathbb{R}^4)$$

- Kepler (celestial mechanics - $T^2_{rot\ P} \sim d^3$, $d = d_{min(P,Sun)}$ perihelion):

$$(t, x) \to (\beta t, \alpha x) = (\beta t, \sqrt[3]{\alpha^2} x), \quad \beta^2 = \alpha^3, \beta > 0$$

Lorentz transformations

- Lorentz transformations which fix the origin in $(\mathbb{R}^2_{1,1}, ds^2 = c^2dt^2 - (dx^1)^2)$

$$\begin{aligned} O(1,1) &= \left\{ A = \tfrac{\pm 1}{\sqrt{1-\beta^2}} \left(\begin{smallmatrix} 1 & \pm\beta \\ \beta & \pm 1 \end{smallmatrix}\right) \middle|\ |\beta| < 1 \right\} \\ &= \left\{ A = \pm \left(\begin{smallmatrix} c & \pm s \\ s & \pm c \end{smallmatrix}\right) \middle| \psi \in \mathbb{R} \right\} \end{aligned}$$

for $\beta = \tanh\psi, c = \cosh\psi, s = \sinh\psi$; four connected pieces (the extremal two are orthocronous):

$$\left\{ \left(\begin{smallmatrix} c & s \\ s & c \end{smallmatrix}\right), \left(\begin{smallmatrix} c & -s \\ s & -c \end{smallmatrix}\right), \left(\begin{smallmatrix} -c & s \\ -s & c \end{smallmatrix}\right), \left(\begin{smallmatrix} -c & -s \\ -s & -c \end{smallmatrix}\right) \right\} \text{ containing}$$

$$\left\{ I = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), P = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right), T = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right), PT = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \right\}$$

Lorentz transformations

- Lorentz transformations on $\mathbb{R}^n_{1,n-1}$ which fix the origin in

$$(\mathbb{R}^n_{p,q}, ds^2 = c^2dt^2 - (dx^1)^2 - \cdots - (dx^{n-1})^2), \quad p + q = n$$

form $O(p,q)$ (four connected components !)

4 Complex transformation groups

4.1 Realization of complex Lie groups

Realization of complex Lie groups

- $\mathbb{C}^n$ as real vector space spanned by $\{e_1, \ldots, e_n, ie_1, \ldots, ie_n\}$
- $\Lambda = A + iB \in M_n(\mathbb{C}) \to r(\Lambda) = \left(\begin{smallmatrix} A & B \\ -B & A \end{smallmatrix}\right) \in M_{2n}(\mathbb{R})$
- $\det(r(\Lambda)) = |\det(\Lambda)|^2$
- $r(GL(n,\mathbb{C})) = \{A \in GL_{2n}(\mathbb{R}) \mid AJ = JA\}$, $J = \left(\begin{smallmatrix} 0 & I_n \\ -I_n & 0 \end{smallmatrix}\right)$
- $SL(n,\mathbb{C}) = \{A \in GL(n,\mathbb{C}) \mid \det(A) = 1\}$

Realization of complex Lie groups

- Realization: $z = x + iy \equiv (x,y)$, $w = u + iv \equiv (u,v)$; inner products:

$$\langle z, w\rangle_{\mathbb{C}} = \sum z^i \bar{w}^i, \quad \langle z, w\rangle_{\mathbb{R}} = \sum z^i u^i + \sum y^i v^i$$

and realization implies $\Re\langle z, w\rangle_{\mathbb{C}} = \langle z, w\rangle_{\mathbb{R}}$.

- $U(n) = \{\Lambda \in GL(n,\mathbb{C}) \mid \langle \Lambda z, \Lambda w\rangle_{\mathbb{C}} = \langle z, w\rangle_{\mathbb{C}}\} = \{\Lambda \in GL(n,\mathbb{C}) \mid \bar{\Lambda}^t\Lambda = I_n\}$
- $r(U(n)) = SO(2n) \cap r(GL(n,\mathbb{C}))$
- similarly in $(\mathbb{C}^n_{p,q}, \langle z, w\rangle_{\mathbb{C}} = \sum_{i=1}^p z^i\bar{w}^i - \sum_{i=p+1}^n z^i\bar{w}^i)$, the metric is preserved by $U(p,q) \supset SU(p,q)$

Particular cases

- $n = 1$: $GL(1, \mathbb{C}) \sim (\mathbb{C}^*, \cdot)$; $r(U(1)) = SO(2)$
- $n = 2$: $SL_2(\mathbb{C}) = \{A = \left(\begin{smallmatrix} a b \\ c d \end{smallmatrix}\right) \mid \det(A) = 1\}$ admits an epimorphism to linear fractional transformations $L = \{\varphi_A : \mathbb{C} \to \mathbb{C} \mid \varphi_A(z) = \frac{az+b}{cz+d}\}$ of kernel $\mathbb{Z}_2$. It follows $L \sim SL(n, \mathbb{C})/\mathbb{Z}_2$
- $U(2) = \left\{ A = \left(\begin{smallmatrix} a & b \\ -\bar{b} & \bar{a} \end{smallmatrix} \right) \;\middle|\; |a|^2 + |b|^2 = 1 \right\}$
- $U(1,1) = \left\{ A = \left(\begin{smallmatrix} a & b \\ \bar{b} & \bar{a} \end{smallmatrix} \right) \mid |a|^2 - |b|^2 = 1 \right\}$
- The latter one admits an injection - not covering $\left\{ \left(\begin{smallmatrix} 0 & d \\ -\bar{d} & 0 \end{smallmatrix} \right) \middle| d \in \mathbb{C} \right\}$,

$$\left(\begin{smallmatrix} c & d \\ \bar{d} & \bar{c} \end{smallmatrix} \right) \in SU(1,1) \to \left(\begin{smallmatrix} 1/c & d/\bar{c} \\ -d/\bar{c} & 1/\bar{c} \end{smallmatrix} \right) = \left(\begin{smallmatrix} a & b \\ -\bar{b} & \bar{a} \end{smallmatrix} \right) \in SU(2)$$

5 Actions, orbits and stabilizers

5.1 Group actions

Group actions

Action

Action of a Lie group G on a manifold M: mapping $\mu : G \times M \to M$ s.t.:
a) μ is differentiable,
b) $\tau_{gg\,\mathrm{pr}} = \tau_g \circ \tau_{g\,\mathrm{pr}}$ (where $\tau_g(x) = \mu(g, x)$) and
c) $\tau_e = Id_M$.

Examples of actions:

- $G \to \mathrm{Diff}\,(G)$ via $L_a(g) = ag,\ R_a(g) = ga,\ j_a(g) = aga^{-1}$
- $GL(n, K) \xrightarrow{\varphi} Diff(P(K^n))$, $\varphi_A(x) = [Ax]$
- linear representations (linear actions): $G \to GL(V)$
- affine actions $G \to \mathrm{Aff}\,(S)$, where S affine space with $\mathrm{Aff}\,(S) = G \circledS T$
- any homomorphism $H \to G$ pull-backs the actions of G on manifolds

5.2 Orbits and stabilizers

Orbits and stabilizers

- *free* action, if $\tau_g x = x \Rightarrow g = e$
- *effective* action, if $\tau_g = Id_M \Rightarrow g = e$
- **orbit** of $x \in M$ is $\mathcal{O}_x = \{\tau_g(x) | g \in G\} \subset M$
- *transitive* action if $\forall x \in M, \mathcal{O}_x = M$
- **stability (isotropy) group** of $x \in M$ is $G_x = \{g \in G | \tau_g(x) = x\}$
- transformation group if $\tau_g \in \mathrm{Diff}\,(M)$.

Example

$M=\mathbb{R}^3, G=SO(3), S_{e_3}=\text{diag}\,(1,SO(2))\sim SO(2)\subset SO(3)$ and

$$\mathcal{O}_{e_3}=\{e_3\}, \mathcal{O}_{e_2}=\{(\cos t,\sin t,0), t\in\mathbb{R}\}$$

Orbits and stabilizers

Remarks

- for $y=gx\in\mathcal{O}_x, G_y=gG_xg^{-1}$ (the stability groups are conjugate along the orbit)
- for $\mu_x: g\to M$ $(x\in M)$, the mapping $\mu_{x*,e}:\mathfrak{g}\equiv T_eG\to T_xM$ maps the right/left invariant vector fields of $\mathfrak{g}$ to $\mathcal{X}(M)$; its image is a Lie subalgebra of $\mathcal{X}(M)$.

Orbits and stabilizers

Theorem

If μ_x has constant rank k, then:
a) G_x is Lie subgroup of G of codimension k and $T_e(G_x)=Ker(\mu_{x*,e})$
b) $\exists U\ni e$ neighborhood in G s.t. $\tau_U(x)\subset M$ is a submanifold of dimension k and $T_x(\tau_U(x))=\mu_{x*,e}T_eG$
c) if $\mathcal{O}_x$ is a submanifold of M then $\dim\mathcal{O}_x=k$.

Symmetric spaces

Theorem

If G acts transitively on M, then $M\sim G/H$, where $H=G_x, x\in M$ **Examples.**

- $S^2\sim SO(3)/SO(2)$
- $\mathbb{R}P^2=SO(3)/(SO(2)\times\mathbb{Z}_2)$
- $CP^n\sim S^{2n+1}/S^1$

6 Symmetric spaces - examples

Symmetric spaces - examples

- real Stieffel manifolds (k-orthonormal frames in $\mathbb{R}^n$) $V^{\mathbb{R}}_{n,k}\sim O(n)/O(n-k)=SO(n)/SO(n-k)$
- complex Stieffel manifolds (k-orthonormal frames in $\mathbb{C}^n$) $V^{\mathbb{C}}_{n,k}\sim U(n)/U(n-k)=SU(n)/SU(n-k)$
- real Grassmann (k-dimensional linear subspaces in $\mathbb{R}^n$) $G^{\mathbb{R}}_{n,k}\sim O(n)/(O(n-k)\times O(k))$
- oriented k linear subspaces in $\mathbb{R}^n$: $\tilde{G}^{\mathbb{R}}_{n,k}\sim SO(n)/(SO(n-k)\times SO(k))$
- complex Grassmann (k-dimensional linear subspaces in $\mathbb{C}^n$) $G^{\mathbb{C}}_{n,k}\sim U(n)/(U(n-k)\times U(k))$
- particular case for $k=1$, $\mathbb{C}P^{n-1}=G^{\mathbb{C}}_{n,1}\sim U(N)/(U(n-1)\times U(1))$

Examples of actions

1. $G=GL(V)\xrightarrow{\varphi}\text{Diff}\,(B_+(V))$, $\varphi_A(b)=A^tbA$ ($M=B_+(V)$ =bilinear symmetric tensors on V, $\dim V=n$). $G_b=O(V,b)$ is a Lie subgroup of G. If b is *non-degenerate*, then O_b is open in M and
$$\dim O(V,b)=\dim GL(V)-\dim\mathcal{O}_b=\frac{n(n-1)}{2}$$
2. $G=GL(V)\xrightarrow{\varphi}\text{Diff}\,(B_-(V))$, $\varphi_A(b)=A^tbA$ ($M=B_-(V)$ =bilinear skew-symmetric tensors on V, $\dim V=n$). $G_b=Sp(V,b)$ is a Lie subgroup of G. If b is *non-degenerate*, then O_b is open in M and
$$\dim Sp(V,b)=\dim GL(V)-\dim\mathcal{O}_b=\frac{n(n+1)}{2}$$
3. $G=GL(V)\xrightarrow{\varphi}\text{Diff}\,(T^1_2(V))$, $(\varphi_A(b))^r_{st}=b^k_{ij}A^i_sA^j_t(A^{-1})^r_k$

7 Infinitesimal isometries

Infinitesimal isometries (Killing vector fields)

- Let (M, g) be an $n-$dimensional (pseudo-)Riemannian manifold. $X \in \mathcal{X}(M)$ is a Killing vector field if its local 1-parameter group is metric-preserving ($\varphi_{t*}g = g, \forall t \in I$)
- X Killing iff $(L_X g) = 0 \Leftrightarrow L_X(g(Y, Z)) = g(L_X Y, Z) + g(Y, L_X Z), \forall Y, Z \in \mathcal{X}(M)$
- For $\{e_i\}_{i=\overline{1,n}}$ local field orthonormal basis, $L_X e_i = a_i^j(x) e_j$ whence X Killing iff $a_{ij} + a_{ji} = 0$ (with lowering via g).

Killing vector fields

- The Killing vector fields form a Lie subalgebra in $\mathcal{X}(M)$ of dimension $m \leq \frac{n(n+1)}{2}$.

Example:

$M = \mathbb{R}^2, g_{ij} = \delta_{ij}$. The isometry group $\mathrm{Iso}\,(\mathbb{R}^2) = O(2) \circledS T$ has its Lie algebra of (maximal) dimension 3, generated by the Killing fields

$$\left\{ \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \right\}$$

8 Groups of automorphisms of geometric structures

Groups of automorphisms of geometric structures

Definition

A subset G of $\mathrm{Diff}\,(M)$ is a Lie transformation group of M iff G is a Lie subgroup of $\mathrm{Diff}\,(M)$ and any flow of G is the flow of an infinitesimal symmetry of $Lie(G)$.

Theorem (Bochner-Montgomery-Zippin, 1955).

G subgroup of $\mathrm{Diff}\,(M)$, where M is a differentiable manifold. If G is locally compact, then G is a Lie transformation group of M.

9 Riemannian isometries

Riemannian isometries

Definition

A map $\varphi : (M, g) \to (N, h)$ between Riemannian manifolds is a Riemannian isometry iff $\varphi^* h = g$ $(h(\varphi_* X, \varphi_* Y) = g(X, Y), \forall X, Y \in \mathcal{X}(M)$.

Theorem (Hopf)

Let (M, g) complete connected Riemannian manifold of constant sectional curvature k. Let $(\tilde{M}, \tilde{g})$ be the standard model of simply connected complete Riemannian space with constant curvature k (hyperbolic for $k < 0$, Euclidean for $k = 0$ and standard sphere of radius $1/\sqrt{k}$, for $k > 0$. Then:
a) There exists a covering projection ("onto" map), smooth local isometry $\varphi : (\tilde{M}, \tilde{g}) \to (M, g)$.
b) If M is simply connected as well, then φ is injective, hence diffeomorphism, and the two manifolds are isometric.

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Finslerian isometries

Definition

A map $\varphi : (\tilde{M}, \tilde{F}) \to (M, F)$ between Finslerian manifolds is a Finslerian isometry iff $\varphi^* F = \tilde{F}$ $(F(\varphi_* X) = \tilde{F}(X), \forall X \in \mathcal{X}(\tilde{M})$.

Theorem (Bao-Chern-Shen)

If two Finsler structures $(\tilde{M}, \tilde{F})$ and (M, F) are positively homogeneous of degree one, and both are connected, $\tilde{F}$ is geodesically complete and there exists a smooth local isometry $\varphi : (\tilde{M}, \tilde{F}) \to (M, F)$, then
a) (M, F) is geodesically complete as well;
b) φ is a surjective covering projection.

Finslerian isometries

Definition

An isometry in (M, F) is a diffeomorphism $\varphi : M \to M$ s.t. $F \circ \varphi_* = F$

Remark

F induces the distance $(d(\varphi(x), \varphi(y)) = d(x, y), \forall x, y \in M$, where $d(x, y) = inf\{l(c)\ c(o) = x, c(1) = y\}$ and $l(c) = \int_0^1 F(\frac{dc}{dt})dt$.

Finslerian isometries

Theorem (Deng-Hou 2002, generalization of the Myers-Steenrod theorem)

Any smooth map $\varphi \in (\mathrm{Diff}\,(M))$ is an isometry on (M, F) iff it is distance-preserving.

Theorem (Deng-Hou 2002)

The group of isometries $G = \mathrm{Iso}\,(M)$ of a Finsler space (M, F) is a Lie transformation group and the isotropy subgroup G_x is compact for all $x \in M$.

Finslerian isometries

Theorem (H.C.Wang 1947)

If (M, F) is a Finsler space of dimension $n \neq 4$, and if $\dim Iso(M) > \frac{n(n-1)}{2} + 1$, then (M, F) is Riemannian.

Theorem (Chao-Hao 1957; Egorov 1974)

There exist Finsler non-Riemannian spaces with $\dim \mathrm{Iso}\,(M) = \frac{n(n-1)}{2} + 1$.

Theorem (L.Kozma 2008)

a) If $||\ \ ||_1, ||\ \ ||_2$ are two Minkowski norms on $\mathbb{R}^n$ and $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ with $||\varphi(a| - \varphi(b)||_2 = ||a - b||_1,\ \forall a, b \in \mathbb{R}^n$, then $\varphi \in \mathrm{Diff}\,(M)$.
b) If φ is a distance-preserving mapping on (M, F), then $\varphi \in \mathrm{Diff}\,(M)$.

Addenda. Invariant 4-tensors in $\mathbb{R}^4$

Theorem [4, 17.1.4]

$T \in \mathcal{T}_4^0(R^4)$ invariant under $SO(4)$ has the form $T_{ijkl} = \lambda\delta_{ik}\delta jl + \mu\delta_{ij}\delta kl + \nu\delta_{il}\delta jk,\ \ (\lambda, \mu, \nu \in \mathbb{R})$

Theorem

$T \in \mathcal{T}_2^2(R^4)$ invariant under $GL_4(\mathbb{R})$ has the form $T^{ij}_{kl} = \lambda\delta^i_k\delta^j_l + \mu\delta^i_l\delta^j_k,\ \ (\lambda, \mu \in \mathbb{R})$

G-structures associated to tensors

Theorem [8]

Let K be a tensor over $\mathbb{R}^n$ and G the subgroup of $GL(n,R)$ which invariates K. Let P be a $G-$structure on $M=\mathbb{R}^n$ (a subbundle of the bundle of frames $L(M)$ with fibre G) and let T be the tensor field induced on M by K and P. Then P is integrable (locally the canonic basis of TM provides frames in P) iff there exist coordinates in M on which T has constant components.

Theorem

If a $G-$structure is integrable, then it admits a torsionless connection

$G-$structures

Examples of $G-$structures

- $G=GL(n,\mathbb{R})$. Any smooth manifold admits the $G-$structure $P=L(M)$, integrable. Any diffeomorphism of M provides an automorphism of P and any vector field on M is an infinitesimal automorphism of P
- $G=GL^+(n,\mathbb{R})$. Any *orientable* smooth manifold admits a $G-$structure; this is integrable. Any orientation-preserving diffeomorphism of M provides an automorphism of P and any vector field on M with orientation-preserving groups with 1 parameter is an infinitesimal automorphism of P

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Exam quizzes

Consider the matrix $A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} \in GL_3(\mathbb{R})$.

I. a) Check that all the principal (Jacobi) minors of A are nontrivial.

b) Apply the *Doolittle, LDU and Crout decompositions* for A towards the upper/lower Heisenberg Lie groups.

c) Using the first decomposition and backward substitution, solve the system $AX = b$, where $X = (x, y, z)^T$ and $b = (7, 8, 9)^T$.

II. a) For the matrix A from above, check that $\det A \neq 0$.

b) Apply *the QR decomposition* for A towards $SO(3)$ and the upper Heisenberg Lie groups.

c) Using this decomposition, the orthogonality of Q and the backward substitution, solve the system $AX = b$, where $X = (x, y, z)^T$ and $b = (7, 8, 9)^T$.

III. Check the Lie algebra properties for the examples provided on page 1 of the lectures.

IV. Check the homomorphism properties for the examples provided on page 2 of the lectures.

V. Check the Lie algebra properties for the examples provided on page 3 of the lectures.

VI. Show that the inversion defined on page 4 of the lectures is self-dual.

VII. Check the consequence of the first realization on page 5 of the lectures.

VIII. Define a Finsler isometry. State several related results.

Seminar addenda I. The LU decomposition algorithm.

1. Consider as input the matrix $A \in GL_n(\mathbb{R})$. Check that the Jacobi minors

$$\Delta_k \equiv \det A^{[k]} = \begin{vmatrix} a_{11} & \dots & a_{1k} \\ \vdots & \ddots & \vdots \\ a_{k1} & \dots & a_{kk} \end{vmatrix},$$

of A are all nonzero. In affirmative case, the algorithm continues.

2. Denote $A = A_1$ and subsequently build the matrices $L_k = [e_1, \dots, e_{k-1}, m_k, e_{k+1}, \dots, e_n]$ and $A_{k+1} = L_k A_k$ $(k = \overline{1, n-1})$, where $A_k = (\tilde{a}_{ij})_{i,j=\overline{1,n}}$ and

$$\begin{cases} e_j = (0, \dots, 0, \underbrace{1}_{j-\text{th}}, 0, \dots, 0)^T, \quad j = \overline{1, n} \\ \\ m_k = \left(0, \dots, 0, \underbrace{1}_{k-\text{th}}, -\dfrac{\tilde{a}_{k+1,k}}{\tilde{a}_{k,k}}, \dots, -\dfrac{\tilde{a}_{n,k}}{\tilde{a}_{k,k}}\right)^T, \quad k = \overline{1, n-1}. \end{cases}$$

3. Denote $L_* = L_1 \cdot \ldots \cdot L_{n-1}$. Check that $L_* \in H_L$ and that $U = A_n$ is upper triangular.

4. Compute $L = L_*^{-1}$.

5. Check that $L \in H_L$ and that $A = L \cdot U$.

Remarks. a) For solving a linear system $AX = b$, use the algorithm to build $L_* \cdot A = U$ and re-write the system equivalently as the triangular system $UX = L_* X$, which can be easily solved by backward substitution, by successively solving and replacing the solutions in the backward sequence of equations $n \to n-1 \to \cdots \to 1$.

b) The algorithm from above provides the Doolittle decomposition, in which $L \in H_L$ and U is upper triangular.

Further, the LDU decomposition is $A = \hat{L}\hat{D}\hat{U}$, with $\hat{L} \in H_L$, $\hat{D} \in \mathbb{D} = \{\text{diag }(d_1, \ldots, d_n) \mid d_k \in \mathbb{R}^*, k = \overline{1,n}\}$ and $\hat{U} \in H_R$, where $\hat{L} = L$, $\hat{D} = \text{diag }(U_{11}, \ldots, U_{nn})$ and $\hat{U} = [U_{11}^{-1}L_1^T, \ldots, U_{nn}^{-1}L_n^T]^T$, where $L_1, \ldots, L_n$ are the lines of the matrix U.

As well, the Crout decomposition provides $A = \bar{L}\bar{U}$, where $\bar{L} = \hat{L}\hat{D}$ is lower triangular and $\bar{R} = \hat{R} \in H_R$.

Seminar addenda II. The QR decomposition algorithm.

1. Consider as input the matrix $A \in GL_n(\mathbb{R})$. Check that $\det A \neq 0$, in which case the algorithm continues.
2. Denote as $u_1, \ldots, u_n$ the column vectors of the matrix A.
3. Apply the Gram-Schmidt orthogonalization process:

$$\begin{cases} v_1 = u_1 \\ v_2 = u_2 - \text{pr}_{v_1} u_2 \\ \ldots \\ v_n = u_n - \text{pr}_{v_1} u_n - \cdots - \text{pr}_{v_{n-1}} u_n, \end{cases} \Leftrightarrow \begin{cases} u_1 = v_1 \\ u_2 = c_{12} v_1 + v_2 \\ \ldots \\ u_n = c_{1n} v_1 + \cdots + c_{n-1,n} v_{n-1} + v_n, \end{cases}$$

   where $\text{pr}_{v_k} u_l = c_{kl} \cdot v_k$, $c_{kl} = \dfrac{\langle u_l, v_k \rangle}{\langle v_k, v_k \rangle}$, $l = \overline{2,l}, k = \overline{1, l-1}$.

4. Re-write the last system as $A = VR$, where V is the matrix having the column vectors $v_1, \ldots, v_n$ and

$$R = \begin{pmatrix} 1 & c_{12} & \ldots & c_{1n} \\ 0 & 1 & \ldots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & c_{n-1,n} \\ 0 & 0 & \ldots & 1 \end{pmatrix}.$$

5. Note that the column vectors of the matrix Q form an orthonormal basis B' of $\mathbb{R}^n$. Hence check that the scalar product of pairwise column-vectors of Q is zero, and the norms of the column-vectors are all equal to 1.
6. Decompose $V = QD$, where $Q = [a_1^{-1}v_1, \ldots, a_n^{-1}v_n]$, $D = \text{diag }(a_1, \ldots, a_n)$ and $a_k = ||v_k||_2$, $k = \overline{1,n}$.
7. Note that the orthogonality of $Q \in O(n)$ infers $\det(Q) \in \{\pm 1\}$. In case that $\det(Q) = -1$, the orthonormal basis B' is negative oriented. In this case transform the B' to a positive-oriented orthonormal basis of $\mathbb{R}^n$, in two possible ways: either replace both the first column of Q and D with their opposites; or interchange two columns of Q and interchange as well the corresponding norms from the diagonal of the matrix D. As result, the column vectors of the new matrix Q will form a *positive oriented* orthonormal basis of $\mathbb{R}^n$ and $Q \in SO(n)$.
8. Point out the QDR decomposition $A = QDR$, with $Q \in SO(n)$, $D \in \mathbb{D}$ and $R \in H_R$.

Remarks. a) The algorithm provides as well the QR decomposition $A = Q \cdot \tilde{R}$, with Q special orthogonal (rotation) matrix and $\tilde{R} = DR$ upper triangular.

b) For solving a linear system $AX = b \Leftrightarrow Q\tilde{R}X = b$, use the algorithm and the orthogonality of the matrix $Q \in SO(n) \subset O(n)$ (namely, $QQ^T = I_n \Rightarrow Q^{-1} = Q^T$) in order to re-write the system equivalently as $\tilde{R}X = \tilde{b}$, where $\tilde{b} = Q^T b$; this is a triangular system which can be easily solved by backward substitution, by successively solving and replacing the solutions in the backward sequence of equations $n \to n-1 \to \cdots \to 1$.