EINSTEIN EQUATIONS FOR THE HOMOGENEOUS FINSLER PROLONGATION TO TM, WITH BERWALD-MOOR METRIC

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Within the geometrical framework provided by (h,v)-metric structures, an important case is that of the homogeneous prolongation (lift) of a Finsler metric to the tangent bundle TM, constructed by R. Miron. In this case, we perform a study of Einstein equations. A special attention is paid to the Berwald-Moor metric, and to metrics conformally related to it.

Keywords: nonlinear (Ehresmann) connection, d-connection, homogeneous prolongation (lift), Einstein equations, Berwald-Moor metric

MSC2000: 53B40,53C60

1 Introduction

In the paper, we apply Miron's theory of Einstein equations on general metric spaces, for h-v models provided by the homogenous prolongation of a Finsler metric. In this context, we show that Einstein's equations in vacuum are satisfied by the homogeneous prolongation of the Berwald-Moor Finslerian metric tensor.

We also investigate the homogeneous lift of conformally deformed Berwald-Moor metrics.

In Sections 2 and 3, we present the mathematical formalism of h-v metrics, [8], [9], which lies at the base of a theory of gravitational and electromagnetic fields in Finsler spaces.

Finslerian metric tensors $g(x, y)$ on a manifold M, by their dependence on directional variables, actually live on the tangent bundle TM of the respective manifold. Once M is a Finsler manifold, then the Finslerian metric tensor endows TM with a Riemannian structure (which is called a *lift* or *prolongation* of the original Finslerian one on M), and the specific instruments of Riemannian geometry can be applied on TM. This idea, applied to Einstein equations, lies at the base of Miron's formalism, which extends classical Einstein equations, and which is presented in Section 4.

In Finsler spaces, the tangent space $T_{x_0}M$ at a point $x_0 \in M$ is itself a Riemannian manifold, and, generally, it is curved. The geometry of the fibre $T_{x_0}M$ influences on the energy-momentum tensor, and this influence is pointed out by Einstein's equations on TM.

Section 5 is devoted to the homogeneous prolongation (or lift) of a Finsler metric, which was also introduced by R. Miron. Homogeneity and homogeneous prolongations are needed in a theory of geodesics and Jacobi fields on TM, in order to ensure the independence of the distance Lagrangian to (at least a group of) reparametrizations.

In Section 6, we apply the above theories for the 4-dimensional Berwald-Moor space; here, the homogeneous lift provides a much simpler model than the usual Sasaki lift. More precisely, we show that, if we use the idea of homogeneous prolongation together with Berwald-Moor metric and a conveniently chosen linear connection, the energy-momentum tensor on TM identically vanishes (even though the curvature tensor $\mathbb{R}$ on TM is not identically zero). As shown in [2], if we used the Sasaki lift instead the homogeneous one, the vertical Ricci tensor S_{ab} would no longer vanish.

The last section is devoted to deformations by a conformal factor $\sigma(x)$ of the above model.

2 Nonlinear and linear connections on TM

Let M be a differentiable manifold of dimension n and class $\mathcal{C}^{\infty}$, (TM,π,M), its tangent bundle and (x^i, y^a) the coordinates of a point $u\in TM$ in a local chart. We denote by "$_{,i}$" partial (usual) derivation with respect to x^i and by "$\cdot_a$", partial derivation with respect to y^a.

Let TM be endowed with a nonlinear (Ehresmann) connection N, [5], [1], [9], and $(\delta_i, \dot\partial_a)$ be the corresponding adapted basis on TM:

$$\delta_i = \frac{\partial}{\partial x^i} - N^a_{\ i}\frac{\partial}{\partial y^a}, \qquad \dot\partial_a = \frac{\partial}{\partial y^a};$$

analogously, let $(dx^i, \delta y^a)$ be its dual basis,

$$\delta y^a = dy^a + N^a_{\ i} dx^i.$$

If the nonlinear connection N is given, then any vector field $X\in\mathcal{X}(TM)$ is locally represented as

$$X = X^{(0)i}\frac{\delta}{\delta x^i} + X^{(1)i}\frac{\partial}{\partial y^i},$$

with $X^{(0)i}$, $X^{(1)i}$ - *distinguished* (or *d-) vector fields*. In the same manner, a 1-form ω on TM can be uniquely written as

$$\omega = \omega_{(0)i}dx^i + \omega_{(1)i}\delta y^i,$$

where $\omega_{(0)i}$, $\omega_{(1)i}$ are *distinguished 1-forms.*

We adopt the following convention: if no elsewhere specified, indices $i,j,k,...$ will denote the quantities corresponding to horizontal geometrical objects, while a,b,c will index quantities corresponding to the vertical distribution.

A *distinguished linear connection* (or, simply, a *d-connection*), [9], [8], is a linear connection D which preserves by parallelism the distributions generated by the nonlinear connection N, i.e., the covariant derivative of any horizontal vector field remains horizontal, while the covariant derivative of any vertical vector field remains vertical. In local coordinates, a d-connection is characterized by its coefficients $(L^i_{\ jk}, L^a_{\ bk}, C^i_{\ jc}, C^a_{\ bc})$, where:

$$\begin{aligned} D_{\delta_k}\delta_j &= L^i_{\ jk}\delta_i, \quad D_{\delta_k}\dot\partial_b = L^a_{\ bk}\dot\partial_a \\ D_{\dot\partial_c}\delta_j &= C^i_{\ jc}\delta_i, \quad D_{\dot\partial_c}\dot\partial_b = C^a_{\ bc}\dot\partial_a. \end{aligned}$$

We shall denote the local components of the torsion tensor of such a linear connection $\mathbb{T}$ by $T^i_{\ jk}$, $R^a_{\ jk}$, $P^i_{\ jc}, P^a_{kb}$, $S^i_{\ bc}$, $S^a_{\ bc}$ (as in [1], [5], [8]): $h\mathbb{T}\left(\frac{\delta}{\delta x^k}, \frac{\delta}{\delta x^j}\right) = T^i_{\ jk}\frac{\delta}{\delta x^i}$, $h\mathbb{T}\left(\frac{\partial}{\partial y^c}, \frac{\delta}{\delta x^j}\right) = P^i_{\ jc}\frac{\delta}{\delta x^i}$ etc. Then:

$$\begin{aligned} T^i_{\ jk} &= L^i_{\ jk} - L^i_{\ kj}, \quad R^a_{\ jk} = \frac{\delta N^a_j}{\delta x^k} - \frac{\delta N^a_k}{\delta x^j}, \quad P^i_{\ jc} = C^i_{\ jc}, \\ P^a_{\ jb} &= \frac{\partial N^a_{\ j}}{\partial y^b} - L^a_{\ bj}, \; S^i_{\ bc} = 0, \quad S^a_{\ bc} = C^a_{\ bc} - C^a_{\ cb}. \end{aligned}$$

With the same convention of notations of indices, the local components of the curvature are, [1], [5], [8]:

$$\begin{cases} R_j{}^i{}_{kl} = \dfrac{\delta L^i_{\ jk}}{\delta x^l} - \dfrac{\delta L^i_{\ jl}}{\delta x^k} + L^h_{\ jk}L^i_{\ hl} - L^h_{\ jl}L^i_{\ hk} + C^i_{\ ja}R^a_{\ kl}, \\ R_b{}^a{}_{kl} = \dfrac{\delta L^a_{\ bk}}{\delta x^l} - \dfrac{\delta L^a_{\ bl}}{\delta x^k} + L^c_{\ bk}L^a_{\ cl} - L^c_{\ bl}L^a_{\ ck} + C^a_{\ bc}R^c_{\ kl}, \end{cases}$$

$$\begin{cases} P_j{}^i{}_{kc} = \dfrac{\partial L^i{}_{jk}}{\partial y^c} - C^i{}_{jc|k} + C^i{}_{jb} P^b{}_{kc} \\ P_b{}^a{}_{kc} = \dfrac{\partial L^a{}_{bk}}{\partial y^c} - C^a{}_{bc|k} + C^a{}_{bd} P^d{}_{kc}, \end{cases}$$

$$\begin{cases} S_j{}^i{}_{bc} = \dfrac{\partial C^i{}_{jb}}{\partial y^c} - \dfrac{\partial C^i{}_{jc}}{\partial y^b} + C^h{}_{jb} C^i{}_{hc} - C^h{}_{jc} C^i{}_{hb}, \\ S_b{}^a{}_{cd} = \dfrac{\partial C^a{}_{bc}}{\partial y^d} - \dfrac{\partial C^a{}_{bd}}{\partial y^c} + C^f{}_{bc} C^a{}_{fd} - C^f{}_{bd} C^a{}_{fc}, \end{cases}$$

where $|$ denotes the horizontal covariant derivative associated to D.

The associated Ricci tensors are, [8]:

$$R_{jk} = R_j{}^i{}_{ki}, \quad \overset{1}{P}_{bj} = P_b{}^a{}_{ja}, \quad \overset{2}{P}_{jb} = P_j{}^i{}_{ib}, \quad S_{bc} = S_b{}^a{}_{ca}.$$

3 $h - v$ metric structures; metrical d-connections

Definition 1. *([8]): An h-v metric on TM is a structure of the form*

$$G = g_{ij}(x,y) dx^i \otimes dx^j + v_{ab}(x,y) \delta y^a \otimes \delta y^b, \tag{1}$$

where g_{ij} and v_{ab} are (0,2)-type symmetric nondegenerate tensor fields on M.

Let G be an h-v metric on TM.

A d-connection D is *metrical if* $D_X G(Y,Z) = 0$, for any vector fields X, Y, Z on TM.

The *canonical metrical linear connection,* [8], is locally given by

$$\begin{aligned} \overset{c}{L}{}^i{}_{jk} &= \frac{1}{2} g^{ih} \Big(\frac{\delta g_{hj}}{\delta x^k} + \frac{\delta g_{hk}}{\delta x^j} - \frac{\delta g_{jk}}{\delta x^h} \Big), \\ \overset{c}{L}{}^a{}_{bk} &= \frac{\partial N^a_k}{\partial y^b} + \frac{1}{2} v^{ac} \Big(\frac{\delta v_{bc}}{\delta x^k} - \frac{\partial N^d{}_k}{\partial y^b} v_{dc} - \frac{\partial N^d{}_k}{\partial y^c} v_{db} \Big), \\ \overset{c}{C}{}^i{}_{jc} &= \frac{1}{2} g^{ih} \frac{\partial g_{jh}}{\partial y^c}, \\ \overset{c}{C}{}^a{}_{bc} &= \frac{1}{2} v^{ad} \Big(\frac{\partial v_{db}}{\partial y^c} + \frac{\partial v_{dc}}{\partial y^b} - \frac{\partial v_{bc}}{\partial y^d} \Big). \end{aligned} \tag{2}$$

The importance of the above connection is given by:

Theorem 2. *([8]): The set of all distinguished connections compatible to G is given by*

$$\begin{aligned} \bar{L}^i{}_{jk} &= \overset{c}{L}{}^i{}_{jk} + \Omega^{ih}_{rj} X^r_{hk}, \ \bar{C}^i{}_{jc} = \overset{c}{C}{}^i{}_{jc} + \Omega^{ih}_{rj} Y^r_{hc} \\ \bar{L}^a{}_{bk} &= \overset{c}{L}{}^a{}_{bk} + \Omega^{ac}_{db} X^d{}_{ck}, \ \bar{C}^a{}_{bc} = \overset{c}{C}{}^a{}_{bc} + \Omega^{af}_{db} X^d{}_{fc}, \end{aligned}$$

where $\Omega^{ih}_{rj} = \frac{1}{2}(\delta^i_r \delta^h_j - g_{rj} g^{ih})$, $\Omega^{ab}_{cd} = \frac{1}{2}(\delta^a_c \delta^b_d - v_{cd} v^{ab})$ and $X^i{}_{jk}$, $X^a{}_{bk}$, $Y^i{}_{jc}$, $Y^a{}_{bc}$ are arbitrary d-tensor fields.

In particular, if we want to a priori give the torsion tensors $T^i{}_{jk}$ and $S^a{}_{bc}$, then there holds

Theorem 3. *([8]): There uniquely exists a d-connection $D\Gamma(N) = (L^i{}_{jk}, L^a{}_{bk}, C^i{}_{jc}, C^a{}_{bc})$ such that:*

1. *D is compatible to G.*
2. $L^a_{\ bk} = \overset{c}{L}{}^a_{\ bk}$, $C^i_{\ jc} = \overset{c}{C}{}^i_{\ jc}$.
3. *The torsion tensors $T^i_{\ jk}$ and $S^a_{\ bc}$ of D are a priori given.*

   (a) *This connection is given by (2) and*

$$\begin{cases} L^i_{\ jk} = \overset{c}{L}{}^i_{\ jk} + \dfrac{1}{2} g^{ir}(g_{rh}T^h_{\ jk} - g_{jh}T^h_{\ rk} + g_{kh}T^h_{\ jr}) \\ C^a_{\ bc} = \overset{c}{C}{}^a_{\ bc} + \dfrac{1}{2} v^{af}(v_{fd}S^d_{\ bc} - v_{bd}S^d_{\ fc} + v_{cd}S^d_{\ bf}). \end{cases} . \tag{3}$$

We shall use, in the following, the notations

$$\tau^i_{\ jk} = \frac{1}{2} g^{ir}(g_{rh}T^h_{\ jk} - g_{jh}T^h_{\ rk} + g_{kh}T^h_{\ jr}),$$
$$\bar{\tau}^a_{\ bk} = \frac{1}{2} v^{af}(v_{fd}S^d_{\ bc} - v_{bd}S^d_{\ fc} + v_{cd}S^d_{\ bf}).$$

4 Einstein equations on TM

Let TM be endowed with: a nonlinear connection N, an h-v metric structure G and a metrical d-connection D with a priori given torsions $T^i_{\ jk}$ and $S^a_{\ bc}$, as in (3).

Once given an h-v metric G on TM, (TM, G) becomes a Riemannian manifold of dimension $2n$. One can formally state the Einstein equations on TM :

$$Ric\,(D) - \frac{1}{2} Sc\,(D)\, G = \kappa \mathcal{T}.$$

In local coordinates, the above relation becomes:

Theorem 4. *([8]) The Einstein equations of (TM, G) have the following form:*

$$R_{ij} - \frac{1}{2}(R+S)g_{ij} = \kappa \mathbb{T}_{ij}$$
$$\overset{1}{P}_{ai} = \kappa \mathbb{T}_{ai}, \quad \overset{2}{P}_{ia} = -\kappa \mathbb{T}_{ia}$$
$$S_{ab} - \frac{1}{2}(R+S)v_{ab} = \kappa \mathbb{T}_{ab},$$

where $\mathbb{T}_{ij}$, $\mathbb{T}_{ai}$, $\mathbb{T}_{ia}$ and $\mathbb{T}_{ab}$ are the local adapted components of the energy momentum tensor.

Comment: in the above equations, the unknowns are not only the components g_{ij}, v_{ab} of the metric tensor, but also the coefficients $N^a_{\ i}$ and the torsions $T^i_{\ jk}$, $S^a_{\ bc}$, this is, in the most general case, one has

$$n(n+1) + n^2 + n^2(n-1) = n^3 + n^2 + n$$

unknowns. Once we fix the nonlinear connection N, their number decreases with n^2, this is, we still have $n^3 + n$ unknown functions ($n = 4 \Rightarrow 68$ unknown functions!).

If we also choose $T^i_{\ jk}$, $S^a_{\ bc}$ as being 0, this is, if we work with the canonical d-connection $D\Gamma(N)$, the remaining unknown functions are only the components of the metric, this is, $n(n+1) \overset{n=4}{=} 20$ unknowns depending both on x and y. If we also establish a link between g_{ij} and v_{ab}, the remaining unknown functions are only 10, just as in the classical Riemannian case. Also, we can work with a given (known) metric and infer nonlinear connections/torsions out of Einstein equations.

The equations are in number of $4n^2$, this is, for $n = 4$, we have 64 equations. Still, in particular cases, as we shall see, this number can drastically reduce.

Remark 5. *g_{ij}, v_{ab} and $N^a{}_i$ appear in the equations with their second order derivatives, while, for $T^i{}_{jk}$ and $S^a{}_{bc}$, the above system is a PDE system of order one.*

The energy conservation law, [8],

$$div\, \mathbb{T} = 0,$$

in local coordinates takes the following form:

$$\begin{aligned} \{R^i{}_j - \frac{1}{2}(R+S)\delta^i_j\}_{|i} + \overset{1}{P}{}^a{}_j|_a &= 0 \\ \{S^a{}_b - \frac{1}{2}(R+S)\delta^a{}_b\}|_a - \overset{2}{P}{}^i{}_{b|i} &= 0. \end{aligned}$$

5 Einstein equations for the homogeneous prolongation (lift) of a Finsler metric

The notion of homogeneous lift is defined by R. Miron, [7], for Finsler metrics.

Its usefulness is the following: the use of the homogeneous lift insures the invariance of the distance Lagrangian $\int \sqrt{G(x,\dot{x})}dt$ on TM to reparametrizations of the form $t \mapsto \lambda t$, and consequently, the possibility of building an exponential map on TM.

Let g define a Finslerian metric tensor on M.

Definition 6. *The homogeneous prolongation (lift) of the Finsler metric g to the tangent bundle TM is the following (h,v)-metric:*

$$G^H = g_{ij}(x,y)dx^i \otimes dx^j + \alpha \frac{g_{ab}(x,y)}{F^2}(x,y)\delta y^a \otimes \delta y^b, \tag{4}$$

where

- *$F^2 = \|y\|^2 = g_{ij}y^iy^j$, and $\alpha > 0$ is a constant;*
- *$g_{ab} = \delta^i_a \delta^j_b g_{ij}$;*
- *(δy^a) are computed w.r.t. the canonical nonlinear connection*

$$N^a{}_j = \frac{\partial G^a}{\partial y^j}, \quad \mathcal{G}^a = \frac{1}{2}g^{ab}\Big(\frac{\partial g_{00}}{\partial y^b \partial x^k}y^k - \frac{\partial g_{00}}{\partial x^k}\delta^k_b\Big) \tag{5}$$

Definition 7. *The Sasaki lift of the generalized Lagrange metric g to TM is*

$$G = g_{ij}(x,y)dx^i \otimes dx^j + g_{ij}(x,y)\delta y^i \otimes \delta y^j, \tag{6}$$

With respect to the Sasaki lift, let us take into account the Cartan connection $C\Gamma(N)$:

$$\begin{aligned} L^i{}_{jk} &= \frac{1}{2}g^{ih}\Big(\frac{\delta g_{hj}}{\delta x^k} + \frac{\delta g_{hk}}{\delta x^j} - \frac{\delta g_{jk}}{\delta x^h}\Big); \\ C^i{}_{jk} &= \frac{1}{2}g^{ih}\Big(\frac{\partial g_{hj}}{\partial y^k} + \frac{\partial g_{hk}}{\partial y^j} - \frac{\partial g_{jk}}{\partial y^h}\Big) = \frac{1}{2}g^{ih}\frac{\partial g_{hj}}{\partial y^k}. \end{aligned}$$

Now, having in view the homogeneous prolongation (4), let us consider the canonical (Cartan) nonlinear connection N determined by the Finslerian fundamental function F, and the following d-connection $M\Gamma(N)$:

$$\begin{aligned}
L^i{}_{jk} &= \frac{1}{2}g^{ih}\Big(\frac{\delta g_{hj}}{\delta x^k}+\frac{\delta g_{hk}}{\delta x^j}-\frac{\delta g_{jk}}{\delta x^h}\Big);\\
\overset{*}{L}{}^a{}_{bk} &= \frac{1}{2}v^{ad}\Big(\frac{\delta v_{db}}{\delta x^k}+\frac{\delta v_{dk}}{\delta x^b}-\frac{\delta v_{bk}}{\delta x^d}\Big);\\
C^i{}_{jc} &= \frac{1}{2}g^{ih}\frac{\partial g_{hj}}{\partial y^c};\\
\overset{*}{C}{}^i{}_{jk} &= \frac{1}{2}v^{ih}\Big(\frac{\partial v_{hj}}{\partial y^k}+\frac{\partial v_{hk}}{\partial y^j}-\frac{\partial v_{jk}}{\partial y^h}\Big).
\end{aligned}$$

Remark 8. *1. In the expression of the coefficients $\overset{*}{L}{}^a{}_{bk}$ we denoted, for simplicity:*

$$v_{dk}=v_{de}\delta^e_k,\quad x^d=\delta^d_j x^j.$$

2. The above connection is a metrical d-connection on TM.
3. Its coefficients $L^i{}_{jk}$ and $C^i{}_{jc}$ coincide with those of the Cartan connection.

Moreover, taking into account that, with respect to the canonical nonlinear connection N we have $\dfrac{\delta F}{\delta x^i}=0,\ i=1,...,4$, we get:

Proposition 9. *The coefficients of the canonical metrical d-connection $M\Gamma(N)$ are given by:*

$$\begin{aligned}
L^i{}_{jk} &= \overset{*}{L}{}^a{}_{bk}\delta^b_j\delta^i_a\\
\overset{*}{C}{}^a{}_{bc} &= C^a{}_{bc}+B^a{}_{bc},
\end{aligned}$$

where $L^i{}_{jk}$ and $C^i{}_{jk}$ are the coefficients of the Cartan connection $C\Gamma(N)$, $C^a{}_{bc}=C^i{}_{jk}\delta^b_j\delta^i_a\delta^k_c$ and

$$B^a{}_{bc}=\frac{-1}{F^2}\big(\delta^a_b y_c+\delta^a_c y_b-y^a g_{bc}\big).$$

Obviously, $B^i{}_{jk}$ are d-tensors of rank (1,2), and their horizontal covariant derivatives with respect to $M\Gamma(N)$ are:

$$B^a{}_{bc|l}=0,\quad B^a{}_{bc}|_{\ e}=\delta^a_b g_{ce}+\delta^a_c g_{be}-\delta^a_e g_{bc}. \tag{7}$$

The torsion tensors of $M\Gamma(N)$ are:

$$T^i{}_{jk}=0,\quad R^a{}_{jc},\ C^i_{jk},\ P^a_{jb}=N^a{}_{j\cdot b}-L^a{}_{bj},\quad S^a{}_{bc}=0.$$

In the following, we shall also use the property, [1], [8]:

$$P^a_{jb}y^b=0,\quad P^a{}_{jb}y^j=0.$$

The curvature tensors are:

$$\begin{aligned}
&R_j{}^i{}_{kl},\quad \overset{*}{R}_b{}^a{}_{kl}=R_b{}^a{}_{kl}+B^a{}_{be}R^e{}_{kl},\\
&P_j{}^i{}_{kc},\quad \overset{*}{P}_b{}^a{}_{kc}=P_b{}^a{}_{kc}+B^a{}_{bd}P^d{}_{kc},\\
&S_j{}^i{}_{bc},\quad \overset{*}{S}_b{}^a{}_{cd}=\overset{*}{C}{}^a{}_{bc\cdot d}-\overset{*}{C}{}^a{}_{bd\cdot c}+\overset{*}{C}{}^f{}_{bc}\overset{*}{C}{}^a{}_{fd}-\overset{*}{C}{}^f{}_{bd}\overset{*}{C}{}^a{}_{fdc},
\end{aligned}$$

where $R_{b\ kl}^{\ a} = R_{j\ kl}^{\ i}\delta_i^a\delta_b^j$, $P_{b\ kc}^{\ a} = P_{j\ kc}^{\ i}\delta_i^a\delta_b^j$.

Consequently, the Einstein equations for the homogeneous lift G^H are:

$$\begin{aligned} &R_{ij} - \frac{1}{2}(R + \overset{*}{S})g_{ij} = \kappa\mathbb{T}_{ij} \\ &\overset{*}{P}_{ai} = \kappa\mathbb{T}_{ai}, \quad P_{ia} = -\kappa\mathbb{T}_{ia} \\ &\overset{*}{S}_{ab} - \frac{1}{2}(R + S)v_{ab} = \kappa\mathbb{T}_{ab}, \end{aligned}$$

where the Ricci tensors: $R_{ij} = R_{i\ jh}^{\ h}$, $\overset{*}{P}_{ai} = \overset{*}{P}_{a\ ic}^{\ c}$, $P_{ia} = P_{i\ ah}^{\ h}$, $\overset{*}{S}_{ab} = \overset{*}{S}_{a\ bd}^{\ d}$ are computed by means of the above.

6 The homogeneous prolongation of the Finslerian Berwald-Moor Metric

Let,again, $\dim M = 4$ and g_{ij} denote the flag Berwald-Moor metric, [2], [3]:

$$g_{ij}(y) = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}, \quad F = \sqrt[4]{y^1y^2y^3y^4}, \tag{8}$$

this is,

$$(g_{ij}) = \frac{1}{8F^2}\begin{pmatrix} -\dfrac{y^2y^3y^4}{y^1} & y^3y^4 & y^2y^4 & y^2y^3 \\ y^3y^4 & -\dfrac{y^1y^3y^4}{y^2} & y^1y^4 & y^1y^3 \\ y^2y^4 & y^1y^4 & -\dfrac{y^1y^2y^4}{y^3} & y^1y^2 \\ y^2y^3 & y^1y^3 & y^1y^2 & -\dfrac{y^1y^2y^3}{y^4} \end{pmatrix}.$$

Its inverse is given by

$$(g^{ij}) = \frac{2}{F^2}\begin{pmatrix} -(y^1)^2 & y^1y^2 & y^1y^3 & y^1y^4 \\ y^1y^2 & -(y^2)^2 & y^2y^3 & y^2y^4 \\ y^1y^3 & y^2y^3 & -(y^3)^2 & y^3y^4 \\ y^1y^4 & y^2y^4 & y^3y^4 & -(y^4)^2 \end{pmatrix},$$

The canonical nonlinear connection (5) has vanishing coefficients:

$$N_{\ i}^a = 0, \qquad a, i = 1, ..., 4$$

and the homogeneous lift of the above looks this way:

$$G = g_{ij}(y)dx^i \otimes dx^j + v_{ab}(y)dy^a \otimes dy^b, \tag{9}$$

where the vertical part of the metric is

$$v_{ab} = \frac{\alpha}{2F^2}g_{ij}\delta_a^i\delta_b^j, \qquad \alpha > 0. \tag{10}$$

We should mention the simplicity of v_{ab}, since their expressions are rational functions of y^i.

By using the expressions of g_{ij} and v_{ab} in (2), we get

Proposition 10. *1. The coefficients of the canonical d-connection of the homogeneous lift of the Berwald-Moor flag metric have the form*

$$\overset{*}{L}{}^{i}{}_{jk} = \overset{*}{L}{}^{a}{}_{bk} = 0, \tag{11}$$

$$\overset{0}{C}{}^{i}{}_{jc} = \frac{p}{8}\frac{y^i}{y^j y^c}, \quad p = \begin{cases} -\frac{3}{8}, & \text{if} \quad i = j = c\,; \\ \frac{1}{8}, & \text{if} \quad i = j \neq c \quad \text{or} \quad i \neq j = c \quad \text{or} \quad i = c \neq j\,; \\ -\frac{1}{8}, & \text{if} \quad i \neq j \neq c \neq i\,. \end{cases} \tag{12}$$

2. The only nonvanishing coefficients $\overset{*}{C}{}^{a}{}_{bc}$ *are*

$$C^a_{aa} = -\frac{1}{y^a}, \quad a = 1, ..., 4. \tag{13}$$

3. The torsion tensor $\mathbb{T}$ *has only one nonvanishing component, namely*

$$P^i{}_{jc} = \overset{*}{C}{}^{i}{}_{jc} = C^i{}_{jc}.$$

By using the above result in order to compute the curvature tensor of the canonical d-connection, we obtain

Proposition 11. *The curvature tensor of the canonical d-connection attached to the homogeneous prolongation of the Finslerian Berwald-Moor metric has as only nonvanishing components:*

$$S_j{}^i{}_{bc} = dx^i\big(\mathbb{R}(\dot\partial_b, \dot\partial_c)\delta_j\big).$$

Since $S_j{}^i{}_{bc}$ do not appear in the construction of the Ricci tensor on TM, the Ricci tensors and Ricci scalars identically vanish.

Thus, we have proven

Proposition 12. *The homogeneous prolongation of the Finslerian Berwald-Moor metric,*

$$G = g_{ij}(y)dx^i \otimes dx^j + v_{ab}(y)dy^a \otimes dy^b, \tag{14}$$

with

$$g_{ij}(y) = \frac{1}{2}\left(F^2\right)_{\cdot i \cdot j}, \quad v_{ab}(y) = \frac{1}{2F^2}\left(F^2\right)_{\cdot a \cdot b}, \quad F^4 = y^1y^2y^3y^4,$$

is a solution for the Einstein equations in vacuum on the tangent bundle TM :

$$R_{ij} - \frac{1}{2}(R+S)g_{ij} = 0$$
$$\overset{*}{P}_{ai} = 0, \quad P_{ia} = 0$$
$$S_{ab} - \frac{1}{2}(R+S)v_{ab} = 0.$$

Remark 13. *By applying a similar procedure, it follows that the* ***homogeneous lift of the flag BM metric****, namely:*

$$\tilde G = \tilde g_{ij}(y)dx^i \otimes dx^j + \tilde v_{ab}(y)\delta y^a \otimes \delta y^b, \tag{15}$$

where $\tilde{g}_{ij}(y) = \dfrac{1}{12F^2}\dfrac{\partial^2 F^4}{\partial y^i \partial y^j}$, $F = \sqrt[4]{y^1y^2y^3y^4}$, $v_{ab} = \dfrac{\alpha}{12F^4}\dfrac{\partial^2 F^4}{\partial y^a \partial y^b}$, $\alpha > 0$, is also a solution for the Einstein equations in vacuum on TM.

Remark 14. *1. The above results are obtained by using the canonical nonlinear connection and the canonical metrical d-connection. By using a different nonlinear connection N, or a metrical d-connection with torsion, we can also obtain nonvanishing values of the energy-momentum tensor.*

2. If, instead of the homogeneous lift we would have used the Sasaki lift (g, g) of the Finslerian Berwald-Moor metric, i.e.,

$$G = g_{ij}(y)dx^i \otimes dx^j + g_{ab}(y)dy^a \otimes dy^b, \tag{16}$$

then, the vertical Ricci tensor $\overset{*}{S}_{ab}$ would have not vanished, hence the Sasaki lift of the BM flag metric (together with the canonical connections N and D) does **not** *give a solution for Einstein's equations in vacuum.*

7 Homogeneous lifts of metrics conformally related to Berwald-Moor one

Let, for the beginning, (M, F) denote an arbitrary Finsler space. Finsler spaces which are conformally related to (M, F), in the sense of ([1]) (angle-preserving), are described by fundamental functions of the form

$$\tilde{F} = e^{\frac{1}{2}\sigma(x)}F,$$

where σ is a real valued smooth function.

This is, the corresponding metric tensors $g_{ij} = \dfrac{1}{2}\dfrac{\partial^2 F^2}{\partial y^i \partial y^j}$ and $\tilde{g}_{ij} = \dfrac{1}{2}\dfrac{\partial^2 \tilde{F}^2}{\partial y^i \partial y^j}$ are related by

$$\tilde{g}_{ij} = e^{\sigma} g_{ij}.$$

It follows that the homogenized versions

$$v_{ab} = \frac{g_{ab}}{F^2}, \quad \tilde{v}_{ab} = \frac{\tilde{g}_{ab}}{\tilde{F}^2}$$

coincide:

$$v_{ab} = \tilde{v}_{ab}.$$

Hence, there holds:

Proposition 15. *The homogeneous lift of any conformally deformed Finslerian metric $\tilde{F} = e^{\frac{1}{2}\sigma(x)}F$, is given by:*

$$\tilde{G} = e^{\sigma(x)}g_{ij}dx^i \otimes dx^j + \frac{g_{ab}}{F^2}\delta y^a \otimes \delta y^b, \tag{17}$$

where g_{ij} is the metric tensor associated to the "undeformed" Finsler function F.

This is, a conformal factor $\sigma(x)$ actually affects only the horizontal part of the metric.

As a remark, if we had used the Sasaki lift instead of the homogeneous one, the vertical part v_{ab} of the metric would have also been multiplied by e^{σ}.

Let now g_{ij} denote the Berwald-Moor Finslerian metric:

$$g_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}, \quad F = \sqrt[4]{y^1y^2y^3y^4}.$$

Then, geodesics of the conformally deformed model $(M, \tilde{F})$ are described by

$$\frac{d^2x^i}{dt^2} + 2G^i(x, \dot{x}) = 0,$$

where $2G^i(x, y) = \gamma^i{}_{jk} y^j y^k$ and

$$\gamma^i{}_{jk} = \frac{1}{2}\tilde{g}^{ih}(\tilde{g}_{hj,k} + \tilde{g}_{hk,j} - \tilde{g}_{jk,h}).$$

By a direct computation, we get:

$$\gamma^i{}_{jk} = \frac{1}{2}(\delta^i_j \sigma_{,k} + \delta^i_k \sigma_{,j} - g^{ih} g_{jk} \sigma_{,h}).$$

Remark 16. *With the notations in the previous sections, we have, actually,*

$$\gamma^i{}_{jk} = A^{ih}_{jk} \sigma_{,h},$$

where A^{ih}_{jk} depend only on y and $\sigma_{,h}$ only on x.

It follows that

$$2G^i = (2y^i y^h - g^{ih} F^2)\sigma_{,h}.$$

Taking into account the form of the BM contravariant metric tensor, we get that

$$2G^i = 2(y^i)^2 \sigma_{,i}, \quad i = 1, ..., 4,$$

where, in the above, there is no summation over i. We have thus proven:

Proposition 17. *Geodesics of the conformally deformed Berwald-Moor Finsler space $(M, e^{\frac{\sigma(x)}{2}} F)$ are given by:*

$$\begin{aligned} \ddot{x}^1 + 2(\dot{x}^1)^2 \sigma_{,1} &= 0, \quad \ddot{x}^2 + 2(\dot{x}^2)^2 \sigma_{,2} = 0, \\ \ddot{x}^3 + 2(\dot{x}^3)^2 \sigma_{,3} &= 0, \quad \ddot{x}^4 + 2(\dot{x}^4)^2 \sigma_{,4} = 0. \end{aligned}$$

Corollary 18. *The only nonvanishing coefficients of the canonical nonlinear connection (5) given by the conformally deformed Berwald-Moor metric $e^{\sigma(x)} g_{ij}$ are*

$$N^i_i = 2y^i \sigma_{,i}, \quad i = 1, ..., 4$$

(where, again, there is no summation over i).

Then, the coefficients of the canonical connection $M\tilde{\Gamma}(N)$ are given by:

$$\begin{aligned} \tilde{L}^i{}_{jk} &= \overset{*}{\tilde{L}}{}^a{}_{bk} \delta^i_a \delta^b_j = \frac{1}{2}\tilde{g}^{ih}(\tilde{g}_{hj;k} + \tilde{g}_{hk;l} - \tilde{g}_{jk;h}); \\ \tilde{C}^i_{jk} &= C^i{}_{jk}, \quad \tilde{C}^a_{bc} = \overset{*}{C}{}^a{}_{bc}, \end{aligned}$$

where $C^i{}_{jk}, \overset{*}{C}{}^a{}_{bc}$ denote the coefficients of the canonical d-connection attached to the undeformed homogeneous prolongation of the BM metric.

The nonvanishing components of the torsion tensor are $R^a{}_{jk}$, $P^i{}_{jc} = C^i{}_{jc}, P^a_{jb}$.

The curvature components which appear in the expressions of the Ricci tensors are:

$$
\begin{aligned}
&R_{j\ kl}^{\ i} = \frac{\delta \tilde{L}^i_{\ jk}}{\delta x^l} - \frac{\delta \tilde{L}^i_{\ jl}}{\delta x^k} + \tilde{L}^h_{\ jk}\tilde{L}^i_{\ hl} - \tilde{L}^h_{\ jl}\tilde{L}^i_{\ hk} + C^i_{\ ja}R^a_{\ kl},\\
&P_{j\ kc}^{\ i} = \frac{\partial \tilde{L}^i_{\ jk}}{\partial y^c} - C^i_{\ jc|k} + C^i_{\ jb}P^b_{\ kc}, \quad \overset{*}{P}_{b\ kc}^{\ a} = P_{j\ kc}^{\ i}\delta^a_i\delta^j_b + B^a_{\ bd}P^d_{\ kc},\\
&\overset{*}{S}_{b\ cd}^{\ a} = 0 \ \Rightarrow \ \overset{*}{S}_{bc} = 0.
\end{aligned}
$$

We notice the following properties:

$$
\tilde{L}^i_{\ ji} = 2\sigma_{,i}, \quad C^i_{\ ic} = 0, \quad B^a_{\ bd}P^d_{\ ka} = 0.
$$

This is, the mixed Ricci tensors are:

$$
\begin{aligned}
P_{jc} &= -C^i_{\ jc|i} + C^i_{\ jb}P^b_{\ ic},\\
\overset{*}{P}_{cj} &= \tilde{L}^a_{\ cj\cdot a} + C^a_{\ cd}P^d_{\ ja},
\end{aligned}
$$

where $\tilde{L}^a_{\ cj} = \delta^a_i\delta^h_c\tilde{L}^i_{hj}$, $C^a_{cd} = \delta^a_i\delta^h_c C^i_{\ hd}$.

Hence, from the Einstein equations on TM, for the conformally deformed model $(M,\tilde{G})$ there remain:

$$
\begin{aligned}
&R_{ij} - \frac{1}{2}Rg_{ij} = \kappa\mathbb{T}_{ij}\\
&-C^i_{\ jc|i} + C^i_{\ jb}P^b_{\ ic} = \kappa\mathbb{T}_{ai}, \quad \tilde{L}^a_{\ cj\cdot a} + C^a_{\ cd}P^d_{\ ja} = -\kappa\mathbb{T}_{ia},\\
&\frac{1}{2}Rv_{ab} = \kappa\mathbb{T}_{ab}.
\end{aligned}
$$

In vacuum, from $\mathbb{T} = 0$, we get $R = 0$. By replacing into the above equations, we get:

Proposition 19. *The Einstein equations in vacuum for the conformally deformed model $(M,\tilde{G})$ are:*

$$
\begin{cases}
R_{ij} &= 0\\
C^i_{\ jc|i} &= C^i_{\ jb}P^b_{\ ic}\\
\tilde{L}^a_{\ cj\cdot a} &= -C^a_{\ cd}P^d_{\ ja}.
\end{cases}
$$

The first set of equations above involves second order derivatives of σ, while the last two of them are PDE's of order 1, linear in $\sigma = \sigma(x)$.

References

[1] Antonelli, P. L., Ingarden, R. S., Matsumoto, M., *The Theory of Sprays and Finsler Spaces with Applications in Physics and Biology,* Kluwer Acad. Publ., 1993.

[2] Balan, V., Brinzei, N.: *Einstein equations for (h,v)-Berwald-Moor relativistic models*, Balkan Journal of Geometry and Its Applications, 11 (2), 2006, 20–26.

[3] Balan, V., Brinzei, N.: *Berwald-Moor-type (h,v)-relativistic models*, Hypercomplex Numbers in Geometry and Physics, 2 (4)/2005, 107–113.

[4] Balan, V., Brinzei, N., Lebedev, S., *Geodesics, Paths and Jacobi Fields for Berwald-Moor Quartic Metrics*, to appear.

[5] Bao, D., Chern, S. S., Shen, Z, *An Introduction to Riemann-Finsler Geometry* (Graduate Texts in Mathematics; 200), Springer Verlag, 2000.

[6] Lebedev, S. V., *The Generalized Finslerian Metric Tensors*, to appear.

[7] Miron, R., *The homogeneous lift to the tangent bundle of a Finsler metric*, Per. Math. Hungarica, 1999.

[8] R. Miron, M. Anastasiei, *Vector Bundles. Lagrange Spaces. Applications in Relativity Theory*, (in Romanian), Ed. Acad., Bucharest, 1987.

[9] Miron, R., Anastasiei, M., *The Geometry of Lagrange Spaces: Theory and Applications*, Kluwer Acad. Publ. FTPH no. 59, (1994).

[10] Miron R., Atanasiu Gh., *Prolongations of the Riemannian, Finslerian and Lagrangian structures,* Rev. Roum. Math. Pures et Appl., 41, 3–4, 1996, 237–249.