ON THE POSSIBILITY OF THE OMPR EFFECT IN THE SPACE WITH FINSLER GEOMETRY. PART 1.

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The effect of the optic-metrical parametric resonance could provide the possibility to obtain the experimental evidence of the gravitational waves existence. The effect might change, if the geometry of the physical space-time is not Riemannian but Finslerian one. The investigation of this situation is undertaken.

1 Introduction

Let us regard a two-level atom in the strong monochromatic quasi-resonant field. The system of Bloch's equations for the components of the density matrix components is

$$\begin{aligned} \frac{d}{dt}\rho_{22} &= -\gamma\rho_{22} + 2i\alpha_1\cos(\Omega t - k_1 y)(\rho_{21} - \rho_{12}), \\ \left(\frac{\partial}{\partial t} + v\frac{\partial}{dy}\right)\rho_{12} &= -(\gamma_{12} + i\omega)\rho_{12} - 2i\alpha_1\cos(\Omega t - k_1 y)(\rho_{22} - \rho_{11}), \\ \rho_{22} + \rho_{11} &= 1. \end{aligned} \tag{1}$$

Here ρ_{22} and ρ_{11} are the populations of the levels, ρ_{12} and ρ_{21} are the polarization terms, γ and γ_{12} are the longitudinal and transversal decay rates of the atom (since level 1 is the ground level, $\gamma_{12} = \gamma/2$); $\alpha_1 = \frac{\mu E}{\hbar}$ is the Rabi parameter (Rabi frequency) proportional to the intensity of the electromagnetic wave (EMW), μ is the dipole momentum, E is the electric stress, $k_1$is the wave vector of the EMW, v is the atom velocity along the Oy-axis pointing at the Earth, $\gamma \ll \alpha_1$ is the condition of strong field.

Let this atom belong to the saturated space maser that is located in the field of the periodic gravitational wave (GW) emitted by a pulsar or a short-period binary star and propagating anti-parallel to the Ox-axis pointing at the GW-source. The GW acts on the atomic levels, on the maser radiation and on the geometrical location of the atom. In [1] it was shown that the first efect is much smaller than the other two effects. The action of the GW on the monochromatic EMW could be accounted for by the solution of the eikonal equation

$$g^{ik}\frac{\partial\psi}{\partial x^i}\frac{\partial\psi}{\partial x^k} = 0. \tag{2}$$

The motion of the atom could be obtained from the solution of the geodesic equation

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{kl}\frac{dx^k}{ds}\frac{dx^l}{ds} = 0 \tag{3}$$

(and not from the solution of the geodesic declination equation as in the calculations of the displacements of the parts of the laboratory setups, designed for the detection of the GW).

The equations (1–3) are basic for the theory of the optic-metrical parametric resonance (OMPR) effect which could provide the possibility to detect the GW in a principially new way. This new way differs from the 18 ones known before [2] by the fact that it is the zero-order and not the first-order effect in the non-dimensional amplitude of the GW. In papers [3], [4], [5], the interpretation of the possible results of the investigation of the geometrical properties of the physical space-time with the help of the OMPR effect is given. The regular case of the isotropic space-time described in terms of Riemannian geometry is described quantitatively and completely, while the case of an anisotropic space-time described by Finsler geometry is described only qualitatively.

Let us give the result corresponding to Riemann geometry. The weak gravitational field in empty space (far from masses) is described by the linearized Einstein equations. Then, for the corrections to the flat space metric tensor, it suffices the wave equation. In the simplest case of plane waves, it has the form

$$\Big(\frac{\partial^2}{\partial x^2}-\frac{1}{c}\frac{\partial^2}{\partial t^2}\Big)h^k{}_i=0. \tag{4}$$

The solution is the expression [6]

$$h^k{}_i = Re[A^k{}_i\exp(ik_\alpha x^\alpha)] \tag{5}$$

that satisfies the equation if $k_\alpha k^\alpha=0$, i. e. k^α is the light-like vector. Then the metric tensor can be written as

$$g^{ik}=\begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & -1+h\cos\frac{D}{c}(x^0-x^1) & 0\\ 0 & 0 & 0 & -1-h\cos\frac{D}{c}(x^0-x^1)\end{pmatrix} \tag{6}$$

where h is the dimensionless amplitude of the GW, D is the frequency of the GW.

The solution of (2) with regard to (6) shows that the action of the GW causes the phase modulation of the EMW. Since h is very small, the phase modulated EMW can be presented as a superposition [7]

$$E(t)=E\cos(\Omega t-ky)+E\frac{\omega}{4D}h[\cos((\Omega-D)t-ky)-\cos((\Omega+D)t-ky)]. \tag{7}$$

The solution of (3) with regard to (6) gives [1]

$$y(t)\sim h\frac{c}{D}\sin(Dt+k_g x) \tag{8}$$

where k_g is the GW wave vector. The expression (8) makes it possible to get the component of the atom velocity directed towards the Earth

$$\begin{aligned} v &= v_0+v_1\cos Dt\,,\\ v_1 &= hc\,. \end{aligned} \tag{9}$$

By substituting (9) and (7) into (1), one gets

$$\begin{aligned}
\frac{d}{dt}\rho_{22} &= -\gamma\rho_{22} + 2i[\alpha_1\cos(\Omega t - ky) + \alpha_2\cos((\Omega - D)t - ky) && (10)\\
&\quad -\alpha_2\cos((\Omega + D)t - ky)](\rho_{21} - \rho_{12}), && (11)\\
\frac{d}{dt}\rho_{12} &= -(\gamma_{12} + i\omega)\rho_{12} - 2i[\alpha_1\cos(\Omega t - ky) + \alpha_2\cos((\Omega - D)t - ky)\\
&\quad -\alpha_2\cos((\Omega + D)t - ky)](\rho_{22} - \rho_{11}), && (12)\\
\rho_{22} + \rho_{11} &= 1,
\end{aligned}$$

where $\alpha_2 = \frac{\omega h}{4D}\alpha_1$, and (9) was used in the the expression for the full derivative $\frac{d}{dt} = \frac{\partial}{\partial t} + kv$. The solution of the system (10) is performed by the asymptotical extension method, the small parameter being $\varepsilon = \frac{\gamma}{\alpha_1}$ (notice, that $\frac{\alpha_2}{\alpha_1} \sim \varepsilon$ too). If the OMPR conditions

$$\begin{aligned}
\frac{\gamma}{\alpha_1} &= \Gamma\varepsilon; \quad \Gamma = O(1); \quad \varepsilon \ll 1, && (13)\\
\frac{\alpha_2}{\alpha_1} &= \frac{\omega h}{4D} = a\varepsilon; \quad a = O(1); \quad \varepsilon \ll 1, && (14)\\
\frac{kv_1}{\alpha_1} &= \frac{\omega h}{\alpha_1} = \kappa\varepsilon; \quad \kappa = O(1); \quad \varepsilon \ll 1, && (15)\\
(\omega - \Omega + kv_0)^2 + 4\alpha_1^2 &= D^2 + O(\varepsilon) \Rightarrow D \sim 2\alpha_1 && (16)
\end{aligned}$$

are fulfilled, then the principal term of the asymptotic expansion for $Im(\rho_{21})$ which characterizes the scattered radiation energy flow can be calculated explicitly. The effect of the OMPR is that at the frequency shifted by D from the central peak of the EMW (that is from the signal of the space maser), the energy flow has the zero order in the powers of the small parameter of the expansion, i. e. is proportional to ε^0, and has the form

$$Im(\rho_{21}) \sim \frac{\alpha_1}{D}\cos 2Dt + O(\varepsilon) \qquad (17)$$

It means that the energy flow is periodically amplified and attenuated with the frequency of the GW. The OMPR signal (17) may be absent while the regular observations due to the time averaging, but it can be registered either with the help of a gate detector checking only the subsequent half-periods, or with the help of the special statistical processing of the radio telescope signal.

One of the assumptions of the qualitative analysis in [3], [4], [5] was that the Einstein equations in empty space in case of the anisotropic space-time still have the form of the wave equation (4) though its solution might become dependent on the direction. In order to prove this assumption and to obtain the generalized model for the OMPR effect in Finsler space the approach developed in [9], [8] will be used. What is actually to be done in this paper (Part 1) is to find some suitable and simple expressions for the metric in the anisotropic case and use them in the eikonal equation and in the geodesic equation. Besides, in the anisotropic case the classical Riemannian forms of the both mentioned equations might need revising. In the subsequent paper (Part 2), we expect to obtain the solutions of these equations and use them for the calculation of the OMPR effect in Finsler case.

2 The weakly deformed model $(\gamma + \varepsilon, v)$

Let M be a 4-dimensional manifold, and TM its tangent bundle. We denote by (x^i, y^i) the coordinates in a local chart on M and by $\left(\delta_i = \frac{\delta}{\delta x^i}, \dot{\partial}_i = \frac{\partial}{\partial y^i}\right)$ the local adapted basis on TM, adapted to a given nonlinear connection N. In the following Sections y^i will be used for $\dfrac{\partial x^i}{\partial t}$, where t denotes an arbitrary parameter and it must not be mixed with the physical coordinate mentioned in the Introduction.

In the following, by "metric", we shall always mean a *generalized Lagrange metric,* [8], this is, a (0,2)-type tensor $\eta = \eta(x, y)$ with the property $\det(\eta_{ij}) \neq 0, \quad \forall (x, y) \in TM.$

In the following, we shall use (h, v)-metric structures on TM in the form

$$G = g_{ij}(x, y) dx^i \otimes dx^j + v_{ab}(y) \delta y^i \otimes \delta y^j, \tag{18}$$

where the (0,2)-type tensors (g_{ij}) and (v_{ab}) have the property

$$\det\bigl(g_{ij}(x, y)\bigr) \neq 0, \quad \det\bigl(v_{ab}(y)\bigr) \neq 0, \quad \forall y \neq 0.$$

If, in some local chart, g_{ij} or v_{ab} do not depend on the positional variables x^i, then they are called *locally Minkowskian.*

Let:

- $v = v(y)$ be a locally Minkowskian metric (which can be specialized, for instance, as a small deformation $\widetilde{\varepsilon}$, as in [12]– [14];
- $g_{ij}(x, y) = \gamma_{ij}(y) + \varepsilon_{ij}(x, y)$, where
- $\gamma = \gamma(y)$ is a 0-homogeneous locally Minkowskian metric tensor, which shall be called in the following, the *undeformed metric;*
- $\varepsilon = \varepsilon(x, y)$ a small deformation of γ (which is not necessarily invertible as a matrix);

By the supposition that γ and ε are 0-homogeneous w.r.t. y, we get that

$$F^2(x, y) = (\gamma_{ij} + \varepsilon_{ij}) y^i y^j$$

defines a Finslerian function. The Finslerian metric defined by F^2 is, in this case,

$$g^*_{ij} = \frac{1}{2} \frac{\partial^2 F^2}{\partial y^i \partial y^j};$$

We should notice that g^* does not necessarily coincide with g_{ij}, but we still have

$$g^*_{ij} y^i y^j = g_{ij} y^i y^j = F^2.$$

In order to prove the above equality, let us remark that $\phi_{ij} = g_{ij} - g^*_{ij}$ generally depends also on y, but being contracted two times with y, it vanishes: $\phi_{ij} y^i y^j = 0$. We have

$$2g^*_{kl} = (F^2)_{\cdot k \cdot l} = 2\phi_{kl} \underbrace{(g_{ij \cdot k \cdot l} y^i y^j + 2g_{lj \cdot k} y^j + 2g_{kj \cdot l} y^j)} + 2g_{kl}\,.$$

From the 0-homogeneity of $g = \gamma + \varepsilon$ as supposed above, we have $g_{ij \cdot k} y^k = 0$, $g_{kj \cdot l} y^l = 0$ etc. This implies $\phi_{kl} y^k y^l = 0$.

In the following, we shall mean by " , ", partial derivation w.r.t. x and by " $\cdot$ ", partial derivative w.r.t. y.

Assumption: We shall neglect all nonlinear terms in ε and its derivatives: $\varepsilon_{ij} \varepsilon_{kl} \simeq 0$, $\varepsilon_{ij} \varepsilon_{kl,m} \simeq 0$, $\varepsilon_{ij} \varepsilon_{kl \cdot a} \simeq 0$.

Let

$$\varepsilon^i{}_j = \gamma^{il} \varepsilon_{lj}, \qquad \varepsilon^{ij} = \gamma^{jk} \varepsilon^i{}_k.$$

Then, obviously, we have

Proposition 2.1 *If the metric γ is nondegenerate, then the deformed metric $g = \gamma + \varepsilon$ is also nondegenerate and its inverse is given by*

$$g^{ij} = \gamma^{ij} - \varepsilon^{ij}.$$

Indeed, by taking into account the previous assumption, we have $g^{ij}g_{jk} = (\gamma^{ij} - \varepsilon^{ij})(\gamma_{jk} + \varepsilon_{jk}) = \delta^i{}_k - \varepsilon^i{}_k + \varepsilon^i{}_k = \delta^i{}_k$.

As a remark, (ε^{ij}) defined above does **not** denote the inverse of (ε_{ij}), while, by (g^{ij}), (γ^{ij}), (v^{ab}) we mean the inverses of the respective matrices.

Let $g = g(x, y)$ and $v = v(x, y)$ be arbitrary metrics. If $\det(v^*_{ab})$ does not vanish anywhere, where

$$v^*_{ab} = \frac{1}{2}\frac{\partial^2}{\partial y^a \partial y^b}\big(v_{de}y^d y^e\big),$$

then the vertical component v_{ab} defines a *canonical* nonlinear connection on TM, [8]:

$$N^a_{\ i} = \frac{\partial \mathcal{G}^a}{\partial y^b}\delta^b_i, \quad \mathcal{G}^a = \frac{1}{2}v^{*ab}\Big(\frac{\partial v_{00}}{\partial y^b \partial x^k}\delta^k_c y^c - \frac{\partial v_{00}}{\partial x^k}\delta^k_b\Big),$$

where the index 0 means transvection by y; in this case, the *canonical metrical linear connection* is given by:

$$\begin{aligned}
L^i{}_{jk} &= \frac{1}{2}g^{ih}\Big(\frac{\delta g_{hj}}{\delta x^k} + \frac{\delta g_{hk}}{\delta x^j} - \frac{\delta g_{jk}}{\delta x^h}\Big),\\
L^a{}_{bk} &= \frac{\partial N^a_k}{\partial y^b} + \frac{1}{2}v^{ac}\Big(\frac{\delta v_{bc}}{\delta x^k} - \frac{\partial N^d_{\ k}}{\partial y^b}v_{dc} - \frac{\partial N^d_{\ k}}{\partial y^c}v_{db}\Big),\\
C^i{}_{jc} &= \frac{1}{2}g^{ih}\frac{\partial g_{jh}}{\partial y^c},\\
C^a{}_{bc} &= \frac{1}{2}v^{ad}\Big(\frac{\partial v_{db}}{\partial y^c} + \frac{\partial v_{dc}}{\partial y^b} - \frac{\partial v_{bc}}{\partial y^d}\Big).
\end{aligned}$$

In the case when $v_{ab} = v_{ab}(y)$ is a locally Minkowski metric, then the above expressions are much simpler, together with those of the local expressions of torsion and curvature:

Proposition 2.2 *[8]: If the vertical part v_{ab} of the metric (1) is locally Minkowski, then:*
1) $N^a_{\ j} = 0$, $L^i{}_{jk} = \gamma^i{}_{jk}$, $L^a{}_{bk} = 0$;
2) $T^i{}_{jk} = 0$, $R^a{}_{ij} = 0$, $P^a{}_{jb} = 0$, $S^a{}_{bc} = 0$;
3) $R_b{}^a{}_{jk} = 0$, $P_b{}^a{}_{kc} = 0$,
where $\gamma^i{}_{jk}(x, y)$ denote the Christoffel symbols of g.

Now, we are able to determine the coefficients of the canonical linear connection.

By a straightforward computation, taking Proposition (2.2) into account, we get:

$$\begin{cases}
N^j{}_a = 0\\
L^i{}_{jk} = \gamma^{il}(\varepsilon_{lj,k} + \varepsilon_{lk,j} - \varepsilon_{jk,l}) =: \gamma^i{}_{jk}\\
L^a{}_{bk} = 0\\
C^a{}_{bc} = \frac{1}{2}v^{ad}(v_{db\cdot c} + v_{dc\cdot b} - v_{bc\cdot d})\\
C^i{}_{jc} = \overset{0}{C}{}^i{}_{jc} - \varepsilon^i_l \overset{0}{C}{}^l{}_{jc} + \frac{1}{2}\gamma^{il}\varepsilon_{lj\cdot c},
\end{cases}.$$

The only nonvanishing components of the torsion tensor are:

$$P^i{}_{jc} = C^i_{jc}.$$

By a direct computation, we obtain the components of the curvature which appear in the expressions of the Ricci tensors:

$$R_{j\ kl}^{\ i} = r_{j\ kl}^{\ i} = \gamma^i_{jk,l} - \gamma^i_{jl,k} + \gamma^h_{\ jk}\gamma^i_{\ hl} - \gamma^h_{\ jl}\gamma^i_{\ hk}, \tag{19}$$

$$P_{j\ kc}^{\ i} = \frac{1}{2}(\delta^i_s\delta^l_j - \gamma^{il}\gamma_{sj})\gamma^s_{\ lk\cdot c}, \tag{20}$$

$$\begin{aligned} P_{b\ kc}^{\ a} &= 0, \\ S_{b\ cd}^{\ a} &= S_{b\ cd}^{\ a}(v) \end{aligned}$$

where $S_{b\ cd}^{\ a}$ depends only on the vertical part (v_{ab}) of the metric.

Consequently, the Ricci tensors and the Ricci scalars are

$$\begin{aligned} R_{jk} &= r_{jk}, \quad R = r, \\ \overset{1}{P}_{bj} &= P_{b\ ja}^{\ a} = 0, \\ \overset{2}{P}_{jb} &= P_{j\ ib}^{\ i} = \frac{1}{2}(\delta^i_s\delta^l_j - \gamma^{il}\gamma_{sj})\gamma^s_{\ li\cdot b}, \\ S_{bc} &= S_{bc}(v), \quad S = v^{ac}S_{ac}. \end{aligned}$$

In vacuum, the second set of equations for our linearized model consists of identities. We get

Theorem 1. *The Einstein equations in vacuum for the linearized model* $(\gamma + \varepsilon, v)$ *are:*

$$\begin{cases} r_{ij} - \dfrac{1}{2}r\gamma_{ij} = \dfrac{1}{2}S(\gamma_{ij} + \varepsilon_{ij}) \\ (\delta^i_s\delta^l_j - \gamma^{il}\gamma_{sj})\gamma^s_{\ li\cdot b} = 0 \\ S_{ab} - \dfrac{1}{2}(r + S)v_{ab} = 0. \end{cases} \tag{21}$$

Here $\gamma_{ij} = \gamma_{ij}(y)$; $\varepsilon_{ij} = \varepsilon_{ij}(x, y)$; $v_{ab} = v_{ab}(y)$. Thus, we see that in Finsler space the Einstein equations have become more complicated.

3 The case of small vertical component v

In the following, we shall assume for simplicity, that the vertical part v of the metric structure G has vanishing Ricci curvature S_{ab}, and it is small enough such that we can neglect terms in the form rv_{ab}:

$$S_{ab} = 0, \quad rv_{ab} \simeq 0.$$

In this case, the Einstein equations (21) become simply:

$$r_{ij} - \frac{1}{2}r\gamma_{ij} = 0 \tag{22}$$

$$(\delta^i_s\delta^l_j - \gamma^{il}\gamma_{sj})\gamma^s_{\ li\cdot b} = 0 \tag{23}$$

As we easily notice, the first set of equations (22) involves only the x-derivatives of the deformation ε, while the second ones, (23), contain mixed derivatives of second order of ε.

In order to integrate the first equations (22), we apply the same procedure as in the classical Riemannian case: namely, we look for solutions satisfying the *harmonicity conditions*

$$\gamma^{ij}\gamma^h_{\ ij} = 0,$$

which are actually

$$\varepsilon^{i}_{j,i} - \frac{1}{2}\varepsilon_{,j} = 0. \tag{24}$$

(There is no loss of generality by the above assumption: if ε does not satisfy (24), then, by taking $\bar{\varepsilon}_{ij} = \varepsilon_{ij} - \frac{1}{2}v_{ij}\varepsilon$, the new unknown functions $\bar{\varepsilon}$ will obey them).

Consequently, the first set of equations (22) becomes

$$\Box\varepsilon_{ij} = 0; \tag{25}$$

this gives a wave solution

$$\varepsilon_{jh} = Re\big(a_{jh}(y)e^{ik_m(y)x^m}\big), \tag{26}$$

where i denotes the imaginary unit. Thus, it turned out that the intuitive assumption made in [5] is right, moreover, it also turned out that the wave vector is no longer isotropic, but also depends on direction.

By (22) and (24), we infer that the quantities $a_{jh}(y)$ and $k_m(y)$ should obey the algebraic system

$$\begin{cases} \gamma^{hl}k_h k_l = 0 \\ a^{i}{}_{j}k_i = \frac{1}{2}a^{i}{}_{i}k_j \end{cases} \tag{27}$$

Remark 3.1 *In the Riemannian case $\varepsilon = \varepsilon(x)$, the quantities a_{jh} and k_j are constants. Still, if γ depends on the directional variables y^i, then, from the first equations above, we deduce that both a and k depend on y. Really, for the equation*

$$\Box\varepsilon_{ij} = 0;$$

we look for a wave solution

$$\varepsilon_{jh} = Re\big(a_{jh}(y)e^{ik_m(y)x^m}\big). \tag{28}$$

We have

$$\Box\varepsilon_{ij} = \gamma^{hl}\varepsilon_{ij,hl} = -\big(\gamma^{hl}k_h k_l\big)\big(\varepsilon \underbrace{a_{ij}(y)\cos(k_m x^m)}\big),$$

and this has to identically vanish. So, either ε itself is zero, or we must have $\gamma^{hl}k_h k_l = 0$.

By taking into account equations (23), we infer that

Proposition 3.1 *The harmonic wave solutions (28) of the Einstein equations (22, 23) are given by the solutions of the system:*

$$\begin{cases} \gamma^{hl}k_h k_l = 0 \\ a^{i}{}_{j}k_i = \frac{1}{2}a^{i}{}_{i}k_j \\ \big(\frac{1}{2}a^{i}{}_{i}k_j\big)_{.b} = \overset{0}{C}{}^{i}{}_{lb}\left\{2a^{l}{}_{j}k_i - a^{l}{}_{i}k_j\right\}. \end{cases} \tag{29}$$

We see that the amplitude $a^{i}{}_{j}$ and the wave vector k_i now depend on each other.

4 Weak Finslerian perturbations of flat Minkowskian metric

Let us suppose that the dimension of $M = \mathbb{R}^4$ and

- the initial metric is the flat Minkowskian one $\gamma = diag(1, -1, -1, -1)$;
- the vertical metric v is as in Section 3.

In this case, we have $\overset{0}{C}{}^i{}_{lb} = 0$, so the system (29) becomes

$$\begin{cases} \gamma^{hl} k_h k_l = 0 \\ a^i{}_j k_i = \frac{1}{2} a^i{}_i k_j \\ \left(\frac{1}{2} a^i{}_i k_j\right)_{\cdot b} = 0. \end{cases} \tag{30}$$

We can easily see that, if we choose

$$k_2 = -k_1 = \frac{D}{c}, \quad k_3 = k_4 = 0$$

(where D and c are constants with physical meaning), and

$$a^3_3 = -a(y), \quad a^4_4 = a(y), \quad a^i{}_j = 0 \quad \text{for all other } (i, j),$$

then (30) is identically satisfied. We have thus obtained the solution

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 + a(y)\cos\left(\frac{D}{c}(x^1 - x^2)\right) & 0 \\ 0 & 0 & 0 & -1 - a(y)\cos\left(\frac{D}{c}(x^1 - x^2)\right) \end{pmatrix}, \tag{31}$$

where $a(y)$ is (any) scalar 0-homogeneous function, small enough such that $a^2 \simeq 0$.

Remark 4.1 *When $a(y)$ is a constant, this metric reduces to the perturbed Minkowski metric for the empty space. In this case the solutions of the geodesics equation and of the eikonal equation are the known ones, (Section 1).*

5 Weak perturbations of Berwald-Moor metric

Let, again the vertical part (v_{ab}) of the metric G be small, as in Section 3.

For the sake of simplicity of computations, we shall use in the following the *flag Berwald-Moor metric, [2], [11]:*

$$\gamma_{ij} = \frac{1}{12F^2} \frac{\partial^2 F^4}{\partial y^i \partial y^j}, \qquad F = \sqrt[4]{y^1 y^2 y^3 y^4} \tag{32}$$

More precisely, we shall consider as unperturbed metric tensor the following:

$$\gamma = \frac{1}{12F^2} \begin{pmatrix} 0 & y^3y^4 & y^2y^4 & y^2y^3 \\ y^3y^4 & 0 & y^1y^4 & y^1y^3 \\ y^2y^4 & y^1y^4 & 0 & y^1y^2 \\ y^2y^3 & y^1y^3 & y^1y^2 & 0 \end{pmatrix}. \tag{33}$$

The system (29), which gives the harmonic solutions of the Einstein equations, is equivalent to:

$$\begin{cases} \gamma_{hl}k^h k^l = 0 \\ a^i{}_j k^j = \frac{1}{2} a^j{}_j k^i \\ \gamma^{lp}(\frac{1}{2} a^i{}_i k_l)_{\cdot b} = \overset{0}{C}{}^i{}_{lb} \; \{2a^p{}_i k^l - a^l{}_i k^p\}. \end{cases} \tag{34}$$

Let us choose

$$\begin{cases} k^1 = \eta \in \mathbb{R} \backslash \{0\}, \qquad k^2 = k^3 = k^4 = 0, \\ \qquad a^{ij} = \lambda k^i k^j, \qquad \lambda \in (0, \delta) \subset \mathbb{R}. \end{cases}$$

This is, the coefficients k_i in the expression $k_m x^m$ are

$$k_1 = 0, \quad k_j(y) = \gamma_{1j} k^1, \quad j \neq 1;$$

The condition $k_1 = 0$ comes from $k_1 = \gamma_{1l} k^l = \gamma_{11} k^1 = 0$ (and, in this case, it does not imply $k^1 = 0$).

We remark that

$$\begin{cases} a^{11} \neq 0, \quad a^{ij} = 0, \quad \forall (i,j) \neq (1,1) \\ k^i k_i = 0. \end{cases}.$$

We have:

$$a^i{}_j = \lambda k^i k_j, \qquad a_{ij} = \lambda k_i k_j = \lambda \gamma_{1i} \gamma_{1j} \eta^2.$$

By a direct computation, and having in view the equalities:

$$\overset{0}{C}_{11b} = \frac{1}{2} \frac{\partial \gamma_{11}}{\partial y^b} = 0, \quad b = 1, ..., 4,$$

it follows that the system (34) is identically satisfied, and, taking into account the considerations in Section 2, the Einstein equations in vacuum are also satisfied by the perturbed metric $\gamma + \varepsilon$.

We have thus obtained

Theorem 2. *The following perturbation*

$$\varepsilon_{jh}(x, y) = \lambda \gamma_{1j} \gamma_{1h} \eta^2 \cos\{\eta(\gamma_{12} x^2 + \gamma_{13} x^3 + \gamma_{14} x^4)\}, \tag{35}$$

where λ and η are real constants, with $\lambda \in (0, \delta)$, and γ_{ij} are the coefficients of the flag Berwald-Moor metric (32), gives a wave solution of the linearized Einstein equations (21).

Remark 5.1 *The contravariant perturbation (35) has only one nonvanishing component, namely, ε^{11} :*

$$\varepsilon^{11} = \lambda \eta^2 \cos\{\eta(\gamma_{12} x^2 + \gamma_{13} x^3 + \gamma_{14} x^4)\} \neq 0, \quad \varepsilon^{ij} = 0, \quad \forall (i,j) \neq (1,1).$$

This property is very useful when dealing with the eikonal equation we are going to obtain in the next Section; also, it suggests that it would be advantageous to work on the dual space T^*M.

6 Eikonal equation

Let us suppose that a plane wave is described by

$$f = ae^{-i\psi(x,y)}.$$

In the first approximation, the eikonal ψ can be written as

$$\psi = \psi_0 + \frac{\partial\psi}{\partial x^i}dx^i + \frac{\partial\psi}{\partial y^a}dy^a.$$

In terms of adapted basis $\left(\delta_i = \dfrac{\delta}{\delta x^i},\quad \dot{\partial}_a = \dfrac{\partial}{\partial y^a}\right)$, it gives

$$\psi = \psi_0 + \frac{\delta\psi}{\delta x^i}dx^i + \frac{\partial\psi}{\partial y^a}\delta y^a.$$

By denoting

$$\begin{aligned} k_i &= \frac{\delta\psi}{\delta x^i}, \qquad K_a = \frac{\partial\psi}{\partial y^a}, \quad i, a = 1, ...4, \\ k^i &= g^{ij}k_j, \qquad K^a = v^{ab}K_b, \end{aligned}$$

we obtain the following *wave vector,* which is globally defined on the tangent bundle TM :

$$K = k^i\delta_i + K^a\dot{\partial}_a.$$

As a remark, the differential $d\psi$ can be written as $d\psi = k_i dx^i + K_a\delta y^a$.

We can formally state the condition that the wave vector should be light-like, namely, $\|K\| = 0$. This has the form

$$g_{ij}k^ik^j + v_{ab}K^aK^b = 0,$$

equivalently,

$$g^{ij}k_ik_j + v^{ab}K_aK_b = 0. \tag{36}$$

The last expression will be called in the following, the *generalized (extended) eikonal equation.*

In the Riemannian case, when $\psi = \psi(x)$, the vertical part K_a of the wave vector vanishes, and the eikonal equation reduces to the classical one:

$$g^{ij}\frac{\partial\psi}{\partial x^i}\frac{\partial\psi}{\partial x^j} = 0.$$

For the (35) the light-like vectors $\|K\| = 0$, are described by the equation

$$\gamma_{ij}k^ik^j + \lambda\eta^2\cos\{\eta(\gamma_{12}x^2 + \gamma_{13}x^3 + \gamma_{14}x^4)\}(k_1)^2 + v_{ij}K^iK^j = 0,$$

where $k_1 = \gamma_{1j}k^j$.

The associated generalized eikonal equation has the form

$$\gamma^{ij}\frac{\partial\psi}{\partial x^i}\frac{\partial\psi}{\partial x^j} = \varepsilon^{11}(x)\left(\frac{\partial\psi}{\partial x^1}\right)^2 - v^{ij}(y)\frac{\partial\psi}{\partial y^i}\frac{\partial\psi}{\partial y^j}.$$

7 Geodesics of perturbed locally Minkovsian metric $g = \gamma + \varepsilon$

The Finslerian function F corresponding to the perturbed B-M metric, namely

$$F^2 = \big(\gamma_{hl}(y) + \varepsilon_{hl}(x, y)\big) y^h y^l$$

leads to the Euler-Lagrange equations

$$\frac{\partial F^2}{\partial x^i} - \frac{d}{dt}\Big(\frac{\partial F^2}{\partial y^i}\Big) = 0, \tag{37}$$

which are equivalent to

$$g^*_{ij}\frac{dy^j}{dt} + \frac{1}{2}\Big(\frac{\partial^2 F^2}{\partial y^i \partial x^j} y^j - \frac{\partial F^2}{\partial x^i}\Big) = 0,$$

where t is the arclength $t = \int\limits_0^t F(x(\tau), y(\tau))d\tau$, $y^i = \dot{x}^i$ and g^*_{ij} is the usual Finsler metric $g^*_{ij} = \dfrac{1}{2}\dfrac{\partial^2 F^2}{\partial y^i \partial y^j}$. As shown in Section 1, we have $g^*_{ij} = \phi_{ij} + g_{ij}$, with

$$\phi_{ij} = \frac{1}{2}(g_{hl\cdot i\cdot j} y^h y^l + 2g_{il\cdot j} y^l + 2g_{jl\cdot i} y^l) \tag{38}$$

The second term, namely, $G_i := \dfrac{1}{2}\Big(\dfrac{\partial^2 F^2}{\partial y^i \partial x^j} y^j - \dfrac{\partial F^2}{\partial x^i}\Big)$ is actually

$$G_i = \gamma_{ilh} y^l y^h + \frac{1}{2}\varepsilon_{hl\cdot i,j} y^h y^l y^j,$$

where $\gamma_{ilh} = \frac{1}{2}(\varepsilon_{il,h} + \varepsilon_{ih,l} - \varepsilon_{lh,i})$.

The equations of geodesics have the form

$$g^*_{ij}\frac{dy^j}{dt} + \gamma_{i00} + \frac{1}{2}\varepsilon_{hl\cdot i,j} y^h y^l y^j = 0, \tag{39}$$

where:

- $\gamma_{i00} = \gamma_{ijk} y^j y^k$;
- $g^*_{ij} = \dfrac{1}{2}\dfrac{\partial F^2}{\partial y^i \partial y^j}$ is the (usual) Finsler metric generated by F;
- the third term originates from the anisotropic deformation of the metric.

8 Discussion

The goal of this paper was to adjust the ideas underlying the theory of the OMPR effect to the case when the geometry used for the description of the space-time is not Riemannian but the Finslerian one. To do this we had first of all to make sure that the linearized Einstein equations in empty space preserve the form of the wave equation for the metrics depending on the direction. Such metric being found had to be used in the eikonal equation and in the geodesic equation. Besides, the very forms of these equations might appear different from those known for the Riemannian case.

The obtained results are the following. The Einstein equations in vacuum for the linearized model (21) were constructed. They do produce the wave equation (25) with the solution given by (28). The last expression differs from the regular plane wave: its amplitude and wave vector are no longer isotropic and depend on each other. If the unperturbed metric tensor is the Minkowski one, then the anisotropic perturbation leads to (31) whose structure is the same as that of (6) common for the GW investigations in Riemannian geometry. If the unperturbed metric is the Berwald-Moor one (33), the structure of the wave solution originating from the linear perturbation of the metric is more complicated and has the form (35). In both these cases the eikonal equation should be generalized and take the form (36), while the geodesic equation has the form (39). It should be also mentioned here that dealing with Finsler geometry, one should consider the revision of Maxwell equations [9] and the testable physical consequences of this. It is very interesting but separate question, because the Bloch's equations needed for the discussion of the OMPR and containing the EMW are one-dimensional, that is they deal only with one direction connecting the space maser and the Earth.

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