THE LAGRANGIAN-HAMILTONIAN FORMALISM IN GAUGE COMPLEX FIELD THEORIES

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An introduction in the study of gauge field theory in terms of complex Finsler geometry on the total space of a G-complex vector bundle E was made by us in [Mu2]. Here we briefly recal the obtained results and similar notions are investigated on the dual bundle E^* by complex Legendre transformation (the $\mathcal{L}$-dual process).

The complex field equations are determined with respect to a gauge complex vertical connections. The complex Hamilton equations are write for the general $\mathcal{L}$-dual Hamiltonian obtained as a sum of particle Hamiltonian, Yang-Mills and Hilbert-Einstein Hamiltonians.

1 Introduction

Gauge theory is called to use the differential geometric methods in order to describe the interactions of fields over a certain symmetry group G.

For initial Yang-Mills gauge theory the Lagrangians had strict local gauge symmetry. After introducing the spontaneously symmetry breaking and Higgs mechanism usually the gauge group is of complex matrices and the gauge Lagrangians are defined over a complexified G-bundle, for instance the Klein-Gordon Lagrangian, Higgs particle Lagrangian or complex fermion-gravitation, etc.. These Lagrangians act on the first order jet manifold, which plays the role of a finite dimensional configuration space of fields. By Legendre morphism, intrinsically related to a Lagrange manifold is the multimomentum Hamiltonian ([Ar,Sa]...) which works on the corresponding phase manifold (the dual G-bundle). Although in Quantum Mechanics the Lagrangian and Hamiltonian formalism is a usual technique, in the gauge field theory it remains almost unknown, especially for the complex situation.

In the present paper, our goal is to introduce a gauge complex field theory in terms of complex Lagrange and Hamilton geometries, [Mu3], extended to an associated fiber of one complex bundle and respectively to its dual bundle.

In the first section, we briefly introduce the geometric machinery which characterize these geometries and then we study the gauge invariance of the main geometric presented objects.

In the next section we recall from [Mu2] the basic notions concerning the complex Euler-Lagrange field equations and the complex gauge invariant Lagrangian for field particle, complex Yang-Mills and Hilbert-Einstein Lagrangians are also written. In the final we translate by complex Legendre transformation the studied results on the dual bundle, and thus we obtain the complex Hamilton field equations and the $\mathcal{L}-$dual Hamiltonians.

2 The geometric background

In [Mu3], we make an exhaustive study of complex Lagrange (particularly Finsler) and Hamilton (Cartan) spaces, which have as a base manifold the holomorphic tangent respectively cotangent bundles of a complex manifold M.

Part of the notions studied in this book can extend to a $G-$complex vector bundle, and here we do this. By this way, since the extension is natural, we will omit the proofs. For more details in this part see the introductory paper [Mu2].

Let M be a complex manifold, $(z^k)_{k=\overline{1,n}}$ complex coordinates in a local chart $(U_\alpha, \varphi_\alpha)$, $\pi : E \to M$ a complex vector bundle of $\mathbb{C}^m$ fiber, and $\eta = \eta^a s_a$ a local section on E, $a = \overline{1,m}$. Consider G a closed m-dimensional Lie group of complex matrices, whose elements are holomorphic functions over M.

Definition 2.1 *A structure of G-complex vector bundle of E is a fibration with transition functions taking values in G.*

This means that if $z'^i = z'^i(z)$ is a local change of charts on M, then the section η changes by the rule

$$z'^i = z'^i(z) \ ; \ \ \eta'^a = M^a_b(z)\eta^b \ , \tag{2.1}$$

where $M^a_b(z) \in G$ and $\partial M^a_b(z)/\partial \bar{z}^k = 0$ for any $a, b = \overline{1,m}$ and $k = \overline{1,n}$.

E has a natural structure of $(n+m)-$complex manifold, a point of E is designed by $u = (z^k, \eta^a)$.

The geometry of E manifold (the total space), endowed with a Hermitian metric $g_{a\bar{b}} = \partial^2 L/\partial\eta^a\partial\bar{\eta}^b$ derived from a homogeneous Lagrangian $L : E \to \mathbb{R}^+$, was intensively studied by T. Aikou ([Ai1,Ai2,Ai3,Mu3]). Let us consider the vertical bundle $VE = \ker \pi^T \subset T'E$. A local base for its sections is $\{\dot{\partial}_a := \frac{\partial}{\partial\eta^a}\}_{a=\overline{1,m}}$ and from (2.1) we have the changes $\dot{\partial}_a = M^b_a(z)\dot{\partial}'_b$. The vertical distribution V_uE is isomorphic to the sections module of E in u.

A supplementary subbundle of VE in $T'E$, i.e. $T'E = VE \oplus HE$, is called a *complex nonlinear connection*, in brief (*c.n.c.*). A local base for the horizontal distribution H_uE, called *adapted* for the (*c.n.c.*), is $\{\delta_k := \frac{\delta}{\delta z^k} = \frac{\partial}{\partial z^k} - N^a_k \frac{\partial}{\partial\eta^a}\}_{k=\overline{1,n}}$, where $N^a_k(z,\eta)$ are the coefficients of the (*c.n.c.*). Locally $\{\delta_k\}$ defines an isomorphism of $\pi^T(T'M)$ with HE if and only if they are changed under the rules $\delta_k = \frac{\partial z'^j}{\partial z^k}\delta'_j$ and hence N^a_k obey a certain rule of transformation.

Definition 2.2 *A gauge complex transformation on G-complex vector bundle E, is a pair $\Upsilon = (F_0, F_1)$, where locally $F_1 : E \to E$ is an F_0-holomorphic isomorphism which satisfies $\pi^T \circ F_1 = F_0 \circ \pi^T$.*

A gauge complex transformation $\Upsilon : u \to \widetilde{u}$ is locally given by a system of analytic functions:

$$\widetilde{z}^i = X^i(z) \ ; \qquad \widetilde{\eta}^a = Y^a(z,\eta) \tag{2.2}$$

with the regularity condition: $\det\left(\frac{\partial X^i}{\partial z^j}\right) \cdot \det\left(\frac{\partial Y^a}{\partial\eta^b}\right) \neq 0$.

Let be $X^i_j := \frac{\partial X^i}{\partial z^j}$ and $Y^a_b := \frac{\partial Y^a}{\partial\eta^b}$; and denote by $X^{\bar{i}}_{\bar{j}}$, $Y^{\bar{a}}_{\bar{b}}$ their conjugates.

Obviously, from the holomorphy requirements we have $X^i_{\bar{j}} = \frac{\partial X^i}{\partial\bar{z}^j} = 0$ and $Y^a_{\bar{j}} = \frac{\partial Y^a}{\partial\bar{z}^j} = 0, Y^a_{\bar{b}} = \frac{\partial Y^a}{\partial\bar{\eta}^b} = 0$.

A (*c.n.c.*) is said to be gauge, (*g.c.n.c*), if the adapted frames transforms into $d-$complex gauge fields, i.e. in addition to $\delta_k = \frac{\partial z'^j}{\partial z^k}\delta'_j$ we have

$$\delta_j = X^i_j\delta_{\widetilde{i}}; \quad \dot{\partial}_b = Y^a_b\dot{\partial}_{\widetilde{a}} \ , \tag{2.3}$$

where $\delta_{\widetilde{i}} = \frac{\delta}{\delta\widetilde{z}^k}$ and $\dot{\partial}_{\widetilde{a}} = \frac{\partial}{\partial\widetilde{\eta}^a}$.

Let us consider now the dual G-bundle $\pi^* : E^* \to M$ of the G-bundle E. Likewise as above, E^* has a natural structure of complex manifold, a point is denoted by $u^* = (z^k, \zeta_a)$, $k = \overline{1,n}$ and $a = \overline{1,m}$, with the following change of charts,

$$z'^i = z'^i(z) \, ; \quad \zeta'_a = \overset{*}{M}{}^b_a(z)\zeta_b \tag{2.4}$$

where $\overset{*}{M}{}_a^b$ is the inverse of M_a^b from (2.1).

By a similar way as for E manifold, we consider $T'E^*$ the holomorphic tangent bundle of E^* and $\{\frac{\partial}{\partial z^k}, \frac{\partial}{\partial \zeta_a}\}$ a base for $T'_{u^*}E^*$. Then $\{\dot{\partial}^a := \frac{\partial}{\partial \zeta_a}\}_{a=\overline{1,m}}$ will be a base for the sections in the vertical bundle $VE^* = \ker \pi^{*T}$ and theret follows the changes $\dot{\partial}^a = \overset{*}{M}{}_b^a \dot{\partial}'^b$. A (*c.n.c*) on E^* is defined by a decomposition $T'E^* = VE^* \oplus HE^*$. The local base for the horizontal distribution $H_{u^*}E^*$ will be denoted by $\{\delta_k^* := \frac{\delta^*}{\delta z^k} = \frac{\partial}{\partial z^k} + N_{ak}\frac{\partial}{\partial \zeta_a}\}_{k=\overline{1,n}}$ and will be called adapted for the (*c.n.c.*) if $\delta_k^* = \frac{\partial z'^j}{\partial z^k}\delta_j^{*\prime}$.

A complex gauge transformation on E^* is defined by a pair $\overset{*}{\Upsilon} = (\overset{*}{F}_0, \overset{*}{F}_1)$, where locally $\overset{*}{F}_1 \colon E^* \to E^*$ is an $\overset{*}{F}_0$-holomorphic isomorphism which satisfies $\pi^{*T} \circ \overset{*}{F}_1 = \overset{*}{F}_0 \circ \pi^{*T}$.

The local expression of a complex gauge transformation on E^* is:

$$\widetilde{z}^i = X^i(z)\ ; \qquad \widetilde{\zeta}_a = Y_a(z, \zeta) \tag{2.5}$$

with the regularity isomorphism condition assumed.

Let be $X_j^i := \frac{\partial X^i}{\partial z^j}$ and $Y_a^b := \frac{\partial Y_a}{\partial \zeta_b}$; then obviously, from the holomorphy requirements, we have $X_{\bar{j}}^i = \frac{\partial X^i}{\partial \bar{z}^j} = 0$ and $Y_{a\bar{j}} = \frac{\partial Y_a}{\partial \bar{z}^j} = 0$; $Y_a^{\bar{b}} = \frac{\partial Y_a}{\partial \bar{\zeta}_b} = 0$.

The various $d-$geometric objects on E^* are defined in complete analogy with those defined by us on E.

A (*c.n.c.*) on E^* is gauge, in brief it is (*g.c.n.c.*), if its adapted frames transform by the rules

$$\delta_j^* = X_j^i \delta_{\tilde{i}}^*\ ;\ \dot{\partial}^a = Y_b^a \dot{\partial}^{\tilde{b}}\ , \tag{2.6}$$

where $\delta_{\tilde{i}}^* = \frac{\delta^*}{\delta \tilde{z}^i}$ and $\dot{\partial}^{\tilde{a}} = \frac{\partial}{\partial \tilde{\zeta}_a}$.

Now, let us consider $L : E \to \mathbb{R}$ a complex regular Lagrangian, that is the function $L(z, \eta)$ defines a metric tensor $g_{a\bar{b}} = \partial^2 L/\partial \eta^a \partial \bar{\eta}^b$ which is Hermitian, $g_{a\bar{b}} = \overline{g_{b\bar{a}}}$ and $\det(g_{a\bar{b}}) \neq 0$ in any point $u = (z, \eta)$ of E. By $g^{\bar{b}a}$ is denoted its inverse metric tensor. The following weighty result was proved in [Mu2]

Proposition 2.1 *If $L(z, \eta)$ is a gauge invariant Lagrangian on E, i.e. $L(z, \eta) = L(\tilde{z}, \tilde{\eta})$, then*

$$N_k^a = g^{\bar{b}a} \frac{\partial^2 L}{\partial z^k \partial \bar{\eta}^b} \tag{2.7}$$

is a (g.c.n.c.).

A fundamental notion in our study is that of d-complex vertical connection on E. The metric tensor $g_{a\bar{b}}$ determines a metric Hermitian structure $\mathbf{G} = g_{a\bar{b}} d\eta^a \otimes d\bar{\eta}^b$ on the vertical bundle VE. The connection form of a d-complex vertical connection D is written according to (7.2.4) from [Mu3] as follows

$$\omega_b^a = L_{bk}^a dz^k + L_{b\bar{k}}^a d\bar{z}^k + C_{bc}^a \delta\eta^c + C_{b\bar{c}}^a \delta\bar{\eta}^c\ , \tag{2.8}$$

where $(dz^k, \delta\eta^c = d\eta^c + N_k^c dz^k)$ is the dual adapted base of the (*c.n.c.*) and $(L_{bk}^a,\ L_{b\bar{k}}^a,\ C_{bc}^a,\ C_{b\bar{c}}^a)$ are the coefficients of the vertical connection D.

From the general theory of Hermitian connection it result a unique metrical Hermitian connection with respect to $\mathbf{G}$ and of $(1,0)$-type, called the *Chern-Lagrange complex connection*, which can be obtained by the same technique as we did for the $T'M$ bundle (Corollary 5.1.1, [Mu3]):

$$\overset{CL}{L}{}_{bk}^a = g^{\bar{d}a} \delta_k g_{b\bar{d}}\ ; \quad \overset{CL}{L}{}_{b\bar{k}}^a = 0\ ; \quad \overset{CL}{C}{}_{bc}^a = g^{\bar{d}a} \dot{\partial}_c g_{b\bar{d}}\ ; \quad \overset{CL}{C}{}_{b\bar{c}}^a = 0. \tag{2.9}$$

A simplification presents a special partial complex connection (cf. [Ai2, Ai3]), called the *complex Bott connection*, which is not metrical but has a very simple expression

$$D_X Y = v\,[X, Y]\ ,\quad \forall X \in HE\ ,\quad Y \in VE.$$

From the calculus of the Lie brackets, see (7.1.10) in [Mu3], it results that the connection form of the complex Bott connection is

$$\omega^a_b = \overset{B}{L^a_{bk}}\, dz^k\ ,\quad \text{where}\quad \overset{B}{L^a_{bk}} = \frac{\partial N^a_k}{\partial \eta^b}.$$

The unique nonzero component of the complex Bott connection on E is

$$\Omega^a_b = R^a_{bi\bar{j}}\, dz^i \wedge d\bar{z}^j \quad \text{with}\quad R^a_{bi\bar{j}} = -\delta_{\bar{j}}\, \overset{B}{L^a_{bi}}\ , \tag{2.10}$$

while the nonzero components of complex Chern-Lagrange connection are more numerous. For this reason the complex Bott connection is an appropriate connection for our approach.

A complex vertical connection determines the following derivative laws on VE :

$$\begin{aligned} \overset{h}{D}_{\delta_k}\, \dot{\partial}_b &= L^a_{bk}\dot{\partial}_a\,; \quad \overset{\bar{h}}{D}_{\delta_{\bar{k}}}\, \dot{\partial}_b = L^a_{b\bar{k}}\dot{\partial}_a\,; \\ \overset{v}{D}_{\dot{\partial}_c}\, \dot{\partial}_b &= C^a_{bc}\dot{\partial}_a\,; \quad \overset{\bar{v}}{D}_{\dot{\partial}_{\bar{c}}}\, \dot{\partial}_b = C^a_{b\bar{c}}\dot{\partial}_a\,. \end{aligned}$$

The covariant derivatives of a vertical field $\Phi = \Phi^a \frac{\partial}{\partial \eta^a}$ will be denoted with $\Phi^a{}_{\shortmid k}$, $\Phi^a{}_{\shortmid \bar{k}}$ and $\Phi^a{}_{|\, b}$, $\Phi^a{}_{|\, \bar{b}}$, where

$$\begin{aligned} \Phi^a{}_{\shortmid k} &= \delta_k \Phi^a + L^a_{bk}\Phi^b\,; \quad \Phi^a{}_{\shortmid \bar{k}} = \delta_{\bar{k}}\Phi^a + L^a_{b\bar{k}}\Phi^b\,; \\ \Phi^a{}_{|\, c} &= \dot{\partial}_c \Phi^a + C^a_{bc}\Phi^b\,; \quad \Phi^a{}_{|\, \bar{c}} = \dot{\partial}_{\bar{c}}\Phi^a + C^a_{b\bar{c}}\Phi^b. \end{aligned} \tag{2.11}$$

If D is a gauge invariant connection, because $\delta_k, \dot{\partial}_c$ and $\delta_{\bar{k}}, \dot{\partial}_{\bar{c}}$ are gauge invariant, we may conclude that these covariant derivatives are gauge invariant as long as Φ is gauge invariant.

On E^* manifold we may introduce the similar d-complex connections with respect to a metric tensor derived from a regular Hamiltonian.

A regular complex Hamiltonian is a real valued function $H : E^* \to \mathbb{R}$ such that $h^{\bar{b}a} = \partial^2 H / \partial\zeta_a \partial\bar{\zeta}_b$ defines a Hermitian metric tensor on E^*, i.e. $\overline{h^{\bar{b}a}} = h^{\bar{a}b}$ and $\det(h^{\bar{b}a}) \neq 0$ on E^*. Let $h_{a\bar{b}}$ be its inverse. A regular complex Hamiltonian determines a metric Hermitian structure on the vertical bundle VE^*, defined by $\mathbf{H} = h^{\bar{b}a}\, d\zeta_a \otimes d\bar{\zeta}_b$. In completly analogy with the result on E we check

Proposition 2.2 *Let $H(z, \zeta)$ be a complex gauge invariant Hamiltonian on E^*, i.e. $H(z, \zeta) = H(\widetilde{z}, \widetilde{\zeta})$. Then,*

$$N_{ak} = -h_{a\bar{b}} \frac{\partial^2 H}{\partial z^k \partial \bar{\zeta}_b} \tag{2.12}$$

is a (g.c.n.c.) on E^*.

With respect to adapted frames of (2.12) (*c.n.c.*) a d-vertical connection on VE^* is denoted by $\overset{*}{D}$ and has the following components,

$$\begin{aligned} \overset{h*}{D}_{\delta^*_k}\, \dot{\partial}^a &= H^a_{bk}\dot{\partial}^b\,; \quad \overset{\bar{h}*}{D}_{\delta^*_{\bar{k}}}\, \dot{\partial}^a = H^a_{b\bar{k}}\dot{\partial}^b\,; \\ \overset{v*}{D}_{\dot{\partial}^c}\, \dot{\partial}^a &= C^{ac}_b\dot{\partial}^b\,; \quad \overset{\bar{v}*}{D}_{\dot{\partial}^{\bar{c}}}\, \dot{\partial}^a = C^{a\bar{c}}_b\dot{\partial}^b \end{aligned}$$

and their conjugates by $\overline{D_X Y} = D_{\bar{X}}\bar{Y}$.

It results that its connection form is

$$\omega^a_b = H^a_{bk} dz^k + H^a_{b\bar{k}} d\bar{z}^k + C^{ac}_b \delta\zeta_c + C^{a\bar{c}}_b \delta\bar{\zeta}_c \ , \tag{2.13}$$

with respect again to the dual adapted frame of the (2.12) (*c.n.c.*).

There exists a unique metric connection with respect to the Hermitian structure **H** on VE^* which is of $(1,0)$-type,

$$\overset{CH}{H^a_{bk}} = h^{\bar{d}a} \delta^*_k h_{b\bar{d}} \, ; \quad \overset{CH}{H^a_{b\bar{k}}} = 0 \, ; \quad \overset{CH}{C^{ac}_b} = -h_{b\bar{d}} \dot{\partial}^c h^{\bar{d}a} \, ; \quad \overset{CH}{C^{a\bar{c}}_b} = 0, \tag{2.14}$$

called the *complex Chern-Hamilton vertical connection.*

A partial vertical *connection of Bott type* on VE^* is given by the vertical part of the bracket, $\overset{*B}{D}_X Y = v\,[X, Y]$, $\forall X \in HE^*$, $Y \in VE^*$, and has the following connection form

$$\omega^a_b = \overset{B}{H^a_{bk}} dz^k \, , \quad \omega^a_b = \overset{B}{L^a_{bk}} dz^k \, , \quad \text{where} \quad \overset{B}{H^a_{bk}} = \frac{\partial N_{bk}}{\partial \zeta_a} \, . \tag{2.15}$$

The unique nonzero component of the complex Bott connection on E^* is

$$\Omega^a_b = \overset{*}{R^a_{bi\bar{j}}} dz^i \wedge d\bar{z}^j \quad \text{with} \quad \overset{*}{R^a_{bi\bar{j}}} = -\delta_{\bar{j}} \overset{B}{H^a_{bi}} \, . \tag{2.16}$$

If H is a gauge invariant Hamiltonian, then both complex Chern-Hamilton and Bott connection on VE^* are gauge invariant. The proof derives from the fact that $h^{\bar{b}a}$ and N_{bk} given by (2.12) are gauge invariant and δ^*_k, $\dot{\partial}^a$ are gauge adapted frames.

The sections of VE^* are 1−forms, $\Phi = \Phi_a(z,\zeta)\frac{\partial}{\partial \zeta_a} = \Phi_a \dot{\partial}^a$. Then a vertical connection $\overset{*}{D}$ on VE^* induces covariant derivatives which act under the section Φ as follows

$$\begin{aligned} \Phi_{a\,\shortmid\,k} &= \delta^*_k \Phi_a + H^b_{ak} \Phi_b \, ; \quad \Phi_{a\,\shortmid\,\bar{k}} = \delta^*_{\bar{k}} \Phi_a + H^b_{a\bar{k}} \Phi_b \, ; \\ \Phi_{a\,|\,^c} &= \dot{\partial}^c \Phi_a - C^{bc}_a \Phi_b \, ; \quad \Phi_{a\,|\,^{\bar{c}}} = \dot{\partial}^{\bar{c}} \Phi_a - C^{b\bar{c}}_a \Phi_b . \end{aligned} \tag{2.17}$$

Now, we recall that in [Mu3] a Lagrangian-Hamiltonian formalism was introduced for the holomorphic tangent bundle $T'M$ by using a complex Legendre morphism. We proved that by complex Legendre transformation (the $\mathcal{L}$-dual process) the image of a complex Lagrange space is (at least locally) a complex Hamilton space. The complex Legendre transformation pushes-forward and its inverse pulls-back the various described geometric objects of a complex Lagrange space and complex Hamilton space, respectively.

Without more other details we can reproduce here, generalizing the $T'M$ case, the process of $\mathcal{L}$-duality for the pairs (E, L) and (E^*, H). Let us consider L a local Lagrangian on $U \subset E$. Then the map $\Lambda : U \subset E \to \bar{U}^* \subset \overline{E^*}$

$$\Lambda : (z^k, \eta^a) \to \left(z^k, \bar{\zeta}_a = \frac{\partial L}{\partial \eta^a}\right) \tag{2.18}$$

is a local diffeomorphism. Since the sections of VE are identified with those of E, we can extend Λ to the open set of VE. By conjugation, the local diffeomorphism $\Lambda \times \overline{\Lambda}$ sends the sections of the complexified bundle $VE \times \overline{VE}$ into sections of $VE^* \times \overline{VE^*}$. This (local) morphism will be called the *complex Legendre transformation*, briefly (*c.L.t*).

Then, locally the function

$$H = \zeta_a \eta^a + \bar{\zeta}_a \bar{\eta}^a - L \tag{2.19}$$

defines a regular (local) Hamiltonian on E^*.

By the inverse $\Lambda^{-1} : \bar{U}^* \to U$, $\Lambda^{-1} : (z^k, \bar{\zeta}_a) \to (z^k, \eta^a = \frac{\partial H}{\partial \zeta_a})$, from a Hamiltonian structure on E^* a Lagrangian structure on E is obtained.

The properties obtained by $(c.L.t)$ are called $\mathcal{L}$-dual one to other. Like in [Mu3], in the following with "$*$" will be designed the image of an object by Λ and with "^{o}" their image by Λ^{-1}. Some of the assertions of § 6.7 from [Mu3] can be easily translated in our framework. For instance, in virtue of (2.19) we have

Proposition 2.3 *The unique pair of (c.n.c.) on VE and respective on VE^* which correspond by $\mathcal{L}$-duality is given by (2.7) and (2.12). Moreover, if L is gauge invariant Lagrangian then both of these (c.n.c.) are gauge invariant.*

Further, simple calculus proves that

Proposition 2.4 *The following equalities hold by $\mathcal{L}$-duality:*

i) $\left(\frac{\delta}{\delta z^k}\right)^* = \frac{\delta^*}{\delta z^k}$; $\left(\frac{\partial}{\partial \eta^a}\right)^* = h_{a\bar{b}} \frac{\partial}{\partial \bar{\zeta}_b}$; $\left(\frac{\delta^*}{\delta z^k}\right)^o = \frac{\delta}{\delta z^k}$; $\left(\frac{\partial}{\partial \zeta_a}\right)^o = g_{a\bar{b}} \frac{\partial}{\partial \bar{\eta}_b}$

ii) $\left(dz^k\right)^* = d^* z^k$; $(\delta \eta^a)^* = h^{\bar{b}a} \delta \bar{\zeta}_b$; $\left(d^* z^k\right)^o = dz^k$; $(\delta \zeta_a)^o = g_{a\bar{b}} \delta \bar{\eta}^b$

iii) $(\mathbf{G})^* = \mathbf{H}$ *and* $(\mathbf{H})^o = \mathbf{G}$.

If D is a metrical connection, then its dual $(D)^*$ is metrical too, moreover their curvatures correspond by $\mathcal{L}$-duality, $(R(X,Y)Z)^* = \overset{*}{R}(X^*, Y^*)Z^*$. We note that the image by $\mathcal{L}$-duality of the complex Bott connection is not the complex Bott connection on E^*. However, we shall use both of these connections for theirs simple expressions and convenience in calculus.

We end this section with a remark. With respect to adapted frames of the $\mathcal{L}$-dual (2.7) and (2.12) $(c.n.c.)$ we can consider the almost simplectic forms ω and θ, $\mathcal{L}$-dual one to other, $\theta = (\omega)^*$,

$$\omega = g_{a\bar{b}}\ \delta \eta^a \wedge \delta \bar{\eta}^b\ ; \quad \theta = h^{\bar{b}a}\ \delta \zeta_a \wedge \delta \bar{\zeta}_b. \tag{2.20}$$

3 The Euler-Lagrange complex field equations

Let E be a G-complex vector bundle over M. From physical point of view a section of E is treated as a field particle. The field particle dynamics assumes to consider the variation of a Lagrangian particle $L_p : E \to \mathbb{R}$, which is a first order differential operator over the sections of E. This is $L_p = L_p(j_1\Phi)$, where $\Phi = \Phi^a s_a$ is a section and $j_1\Phi$ its first jet. Enlarge this is, $\hat{L}_p(\Phi) = L_p(\Phi^a, \partial_i\Phi^a, \partial_{\bar{i}}\Phi^a, \dot{\partial}_b\Phi^a, \dot{\partial}_{\bar{b}}\Phi^a)$ where $\partial_i = \frac{\partial}{\partial z^i}$, $\dot{\partial}_b = \frac{\partial}{\partial \eta^b}$.

The field equations imply to find the particle Φ from the variational principle $\delta\mathcal{A} = \frac{d}{dt}\mid_{t=0} \mathcal{A}(\Phi + t\delta\Phi)$, where $\mathcal{A}(\Phi) = \int \hat{L}_p(\Phi)$ is the action integral. Actually, the action integral is defined on a compact subset $\theta \subset E$ and, for the independence of the integral at the changes of local charts, instead of $\hat{L}_p(\Phi)$ we consider the Lagrangian density $\mathcal{L}_p(\Phi) = \hat{L}_p(\Phi) \mid g \mid^2$, where $\mid g \mid = \mid \det g_{a\bar{b}} \mid$ and $g_{a\bar{b}} = \partial^2 L_p / \partial \eta^a \partial \bar{\eta}^b$ (since L_p depends on (z, η) by means of Φ). In the following the regularity condition for L_p will be assumed.

The problem of solutions for the field equations is one difficult, first because the chosen Lagrangian needs to be one gauge invariant (by means of Φ and its derivatives). Then the derivations in field equations are with respect to the natural frames ∂_i, $\dot{\partial}_b$ which, for a gauge invariant expression of the field equations, need to be replaced with the adapted frames of one $(g.c.n.c.)$, i.e. $\partial_i = \delta_i + N_i^a\ \dot{\partial}_a$. Such a way was followed in [Mu1] in order to obtain the gauge invariant field equations on $T'M$. The modern gauge field theories is based on the "minimal replacement" principle ([Bl, DM, Pa]...), which is nothing but a generalization of Einstein's covariance principle.

The minimal replacement principle consists in replacement in $L_p(\Phi^a, \partial_i\Phi^a, \partial_{\bar{i}}\Phi^a, \dot{\partial}_i\Phi^a, \dot{\partial}_{\bar{i}}\Phi^a)$ partial derivatives with covariant derivatives of a gauge invariant vertical connection, possible the complex Bott connection. At the first glance this seems to be a notational process, but it is a more subtle idea. The connection becomes a dynamical variable which joints mechanics with the geometry of the space. Thus we will study the variation of the action for the Lagrangian $L_p(\Phi, D\Phi)$. But for the beginning let us introduce, as in standard theory, the (complex) currents on E :

$$J(\Phi, D\Phi) \wedge \delta\omega := \frac{d}{dt} |_{t=0} \mathcal{L}(\Phi, D\Phi + t\delta\omega) \tag{3.1}$$

where $\delta\omega$ is a variation for the connection form of D connection.

Direct calculus in (2.2) yields the following complex currents:

$$\overset{h}{J}{}^i_a = \frac{\partial \mathcal{L}}{\partial \Phi^a_{\,\shortmid i}} \; ; \quad \overset{\bar{h}}{J}{}^{\bar{i}}_a = \frac{\partial \mathcal{L}}{\partial \Phi^a_{\,\shortmid \bar{i}}} \; ; \quad \overset{v}{J}{}^b_a = \frac{\partial \mathcal{L}}{\partial \Phi^a_{\,| b}} \; ; \quad \overset{\bar{v}}{J}{}^{\bar{b}}_a = \frac{\partial \mathcal{L}}{\partial \Phi^a_{\,| \bar{b}}} \tag{3.2}$$

which implicitly contain the following components

$$\overset{h}{J}{}^{ib}_a = \frac{\partial \mathcal{L}}{\partial L^a_{bi}} \; ; \quad \overset{\bar{h}}{J}{}^{\bar{i}b}_a = \frac{\partial \mathcal{L}}{\partial L^a_{b\bar{i}}} \; ; \quad \overset{v}{J}{}^{cb}_a = \frac{\partial \mathcal{L}}{\partial C^a_{bc}} \; ; \quad \overset{\bar{v}}{J}{}^{\bar{c}b}_a = \frac{\partial \mathcal{L}}{\partial C^a_{b\bar{c}}} \; .$$

Now, let us focus attention to the variation of the action integral, $\delta\mathcal{A}(\Phi) = \frac{d}{dt} |_{t=0} \int_\theta \mathcal{L}(\Phi, D\Phi + t\delta\omega) = 0$. This implies

$$\int_\theta \left\{ \frac{\partial \mathcal{L}}{\partial \Phi^a}\delta\Phi^a + \frac{\partial \mathcal{L}}{\partial \Phi^a_{\,\shortmid i}}\delta(\Phi^a_{\,\shortmid i}) + \frac{\partial \mathcal{L}}{\partial \Phi^a_{\,\shortmid \bar{i}}}\delta(\Phi^a_{\,\shortmid \bar{i}}) + \frac{\partial \mathcal{L}}{\partial \Phi^a_{\,| b}}\delta(\Phi^a_{\,| b}) + \frac{\partial \mathcal{L}}{\partial \Phi^a_{\,| \bar{b}}}\delta(\Phi^a_{\,| \bar{b}}) \right\} = 0.$$

Further, for instance the calculus of the second term involves

$$\frac{\partial \mathcal{L}}{\partial \Phi^a_{\,\shortmid i}}\delta(\Phi^a_{\,\shortmid i}) = \frac{\partial \mathcal{L}}{\partial \Phi^a_{\,\shortmid i}}\frac{\partial}{\partial z^i}(\delta\Phi^a) + \frac{\partial \mathcal{L}}{\partial \Phi^a_{\,\shortmid i}}\delta(L^a_{bi}\Phi^b) =$$

$$= \frac{\partial}{\partial z^i}\left(\frac{\partial \mathcal{L}}{\partial \Phi^a_{\,\shortmid i}}\delta(\Phi^a)\right) - \frac{\partial}{\partial z^i}\left(\frac{\partial \mathcal{L}}{\partial \Phi^a_{\,\shortmid i}}\right)(\delta\Phi^a) + \frac{\partial \mathcal{L}}{\partial \Phi^a_{\,\shortmid i}}\delta(L^a_{bi}\Phi^b)$$

and analogously for the other terms. If we assume a nul variation on the boundary of θ, then finally for the variation of the integral action we obtain

$$\frac{\partial \mathcal{L}}{\partial \Phi^a} = \frac{\partial}{\partial z^i}(\frac{\partial \mathcal{L}}{\partial \Phi^a_{\,\shortmid i}}) + \frac{\partial}{\partial \bar{z}^i}(\frac{\partial \mathcal{L}}{\partial \Phi^a_{\,\shortmid \bar{i}}}) + \frac{\partial}{\partial \eta^b}(\frac{\partial \mathcal{L}}{\partial \Phi^a_{\,| b}}) + \frac{\partial}{\partial \bar{\eta}^b}(\frac{\partial \mathcal{L}}{\partial \Phi^a_{\,| \bar{b}}}) - < J, \delta\omega > ,$$

where, $$< J, \delta\omega > = \int_\theta \{ \overset{h}{J}{}^i_a \, \delta(L^a_{bi}\Phi^b) + \overset{\bar{h}}{J}{}^i_a \, \delta(L^a_{b\bar{i}}\Phi^b) + \overset{v}{J}{}^c_a \, \delta(C^a_{bc}\Phi^b) + \overset{\bar{v}}{J}{}^{\bar{c}}_a \, \delta(L^a_{b\bar{c}}\Phi^b) \}.$$

Taking into account the (2.3) expressions of the complex currents, in adapted frames of the (2.7) (*c.n.c.*) the previous field equations are written

$$\frac{\partial \mathcal{L}}{\partial \Phi^a} = \delta_i \overset{h}{J}{}^i_a + \delta_{\bar{i}} \overset{\bar{h}}{J}{}^{\bar{i}}_a + \dot{\partial}_b \overset{v}{J}{}^b_a + \dot{\partial}_{\bar{b}} \overset{\bar{v}}{J}{}^{\bar{b}}_a + N^b_i \dot{\partial}_b \overset{h}{J}{}^i_a + N^{\bar{b}}_{\bar{i}} \dot{\partial}_{\bar{b}} \overset{\bar{h}}{J}{}^{\bar{i}}_a - < J, \delta\omega > . \tag{3.3}$$

The (2.4) equations, for $a = \overline{1, m}$, will be called the *complex field equations* of the particle Φ.

The gauge invariance of the Lagrangian L_p, with respect to particle Φ and their covariant derivatives, implies the gauge invariance of the complex currents and consequently the gauge invariance of (2.4) complex field equations. Certainly, everywhere we take in discussion a gauge invariant vertical connection D, particularly the complex Chern-Lagrange or Bott connections.

For the existence of a such gauge invariant particle Lagrangian subsequently we propose a particle Lagrangian of Klein-Gordon type, quite generalized and adequate for various field applications. For this purpose we consider a pair of Hermitian metrics, one being the Lorentz metric $\gamma_{i\bar{j}}(z)$ on the complex world manifold M. The second is a mass Hermitian metric $\gamma_{a\bar{b}}(z,\eta)$ on E, derived from the matter field Lagrangian $L_m = m_{a\bar{b}}\Phi^a\bar{\Phi}^b$ ($m_{a\bar{b}}$ the Hermitian mass matrix). In the last period one Finsler-Minkowski metric kick up some interest in applications of Finsler geometry in relativity, namely the Berwald-Moor metric. We can propose instead of L_m the folowing complex version of Berwald-Moor metric $L_{BM} = \{\prod_a(\eta^a\bar{\eta}^a)\}^{\frac{1}{m}}$. If we wish to connect our field theory with other, a good choose instead of mass metric is one derived from an external Lagrangian with physical meaning, for instance an Antonelly-Shimada complex Lagrangian $L_{AS} = e^{2\sigma(z)}\{\sum_a(\eta^a\bar{\eta}^a)^m\}^{\frac{1}{m}}$ (see [Mu3]), with applications in biology and relativistic optics. However, each of these last complex Lagrangians could be of interest for complex Finsler geometry.

The Hermitian metric $\gamma_{a\bar{b}}$ determines the (2.7) (*c.n.c.*) and its adapted frames. Then, a gauge invariant Lagrangian with respect to a complex vertical connection D and a real valued potential function $V(\Phi)$ can be

$$L_p(\Phi, D\Phi) = \frac{1}{2}\sum_a\{\gamma^{\bar{j}i}D_{\delta_i}\Phi^a D_{\delta_{\bar{j}}}\bar{\Phi}^a + \gamma^{\bar{b}c}D_{\dot{\partial}_c}\Phi^a D_{\dot{\partial}_{\bar{b}}}\bar{\Phi}^a\} + V(\Phi). \tag{3.4}$$

Note that L_p contains informations about matter field by means of $\gamma_{a\bar{b}}$ and by covariant derivatives of the field. $V(\Phi)$ is a potetial fuctions which, for instance, can be considered as beeing $V(\Phi) = \mp m^2 \parallel \Phi \parallel^2 - \frac{1}{4} \parallel \Phi \parallel^4$, with $\parallel \Phi \parallel^2 = \sum_a \Phi^a\bar{\Phi}^a$, for the exact symmetry or for the broken symmetry, respectively.

As we already know from the classical field theory, this particle Lagrangian $L_p(\Phi, D\Phi)$ is not able, quite so in a generalized form, to offer a solid physical theory because it does not contain enough the geometrical aspects of the space (curvature, etc.). For this purpose, in the generalized Maxwell equations the total Lagrangian of electrodynamics is taken in the form:

$$L_e(\Phi, D\Phi) = L_p(\Phi, D\Phi) + L_{YM}(D), \tag{3.5}$$

where

$$L_{YM}(D) = -\frac{1}{2}\Omega \wedge {}^*\Omega \tag{3.6}$$

is a connection Lagrangian, Ω being the curvature form of D and ${}^*\Omega$ is its Hodge dual.

For the complex Bott connection on E we obtain

$$L_{YM}(\overset{B}{D}) = -\frac{1}{2}\sum_{a,b}\gamma^{\bar{j}i}\gamma^{\bar{k}l}R^a_{bi\bar{j}}R^a_{bl\bar{k}}\ .$$

The curvature form of Chern-Lagrange connection is a bit complicate hence we renounce to apply here.

Since, $\delta_D\mathcal{A}_e(\Phi, D\Phi) = \delta_D\mathcal{A}_p(\Phi, D\Phi) + \delta_D\mathcal{A}_{YM}(D)$, and $\delta_D\mathcal{A}_p(\Phi, D\Phi) = -< J, \delta\omega > = - < \delta\omega, {}^* J >$ (${}^*J$ is the dual form current), a computation like in [Pa], yields for the complex Bott connection that $\delta_D\mathcal{A}_{YM}(D) = < \delta\omega, {}^* D^*\Omega >$. Hence, for the complex Bott connection we have that $D^*\Omega = {}^* J$, or else

$$\delta_k\Omega^a_b + L^a_{ck}\Omega^c_b - L^c_{bk}\Omega^a_c = {}^*\overset{h}{J}{}^a_{kb}, \tag{3.7}$$

this generally being called the *complex Yang-Mills equation* on E.

Also we can check that $D^*J = 0$ (the same calculus like for formulae (6.7) from [DM]) and therefore the complex currents are conservative. We note that in this complex Y-M equation the curvature form of Bott connection contains implicitly the Hermitian metric tensor $g_{a\bar{b}} = \partial^2 L_p/\partial\eta^a\partial\bar{\eta}^b$ of the particle Lagrangian.

Finally, for coupling with gravity we again consider the Lorentz Hermitian metric $\gamma_{i\bar{j}}(z)$ on M, which now we assume it derives from a gravitational potential, and $\mathcal{G} = \gamma_{i\bar{j}}dz^i \wedge d\bar{z}^j + g_{a\bar{b}}\delta\eta^a \wedge \delta\bar{\eta}^b$ a metric structure on $T_C E$.

By $S_{i\bar{j}} = \sum S^k_{ki\bar{j}}$ and by $\rho(\gamma) = \gamma^{\bar{j}i}S_{i\bar{j}}$ we denote the Ricci curvature and scalar, respectively, with respect to L-C connection of $\gamma_{i\bar{j}}$ metric lifted on $T_C E$. Also by $R_{i\bar{j}} = \sum R^a_{ai\bar{j}}$ and $\rho(g) = \gamma^{\bar{j}i}R_{i\bar{j}}$ we have the Ricci curvature and scalar, respectively, with respect to Bott connection of the g metric. The sum $\rho = \rho(\gamma) + \rho(g)$ generates an Hilbert-Einstein type Lagrangian $L_G = -\frac{1}{\chi}\rho$, where χ is the universal constant.

The complex Einstein equations on E will be

$$S_{i\bar{j}} - \frac{1}{2}\rho(\gamma)\gamma_{i\bar{j}} = \chi T_{i\bar{j}}\,; \quad R_{i\bar{j}} - \frac{1}{2}\rho(g)\gamma_{i\bar{j}} = \chi T_{i\bar{j}} \tag{3.8}$$

where $T_{i\bar{j}}$ is the stress-energy tensor of the potential gravity $\gamma_{i\bar{j}}(z)$ on M.

The total Lagrangian for coupling gravity with electodynamics (complex inhomogeneus Maxwell equations) is

$$L_t(\Phi, D\Phi) = L_p(\Phi, D\Phi) + L_{YM}(D) + L_G. \tag{3.9}$$

4 Hamiltonian gauge complex theory

In the preview section, in fact a field particle was treated as section $\Phi = \Phi^a(z,\eta)s_a$ on E which induced naturally the section $\Phi = \Phi^a(z,\eta)\dot{\partial}_a$ on VE. The associated particle Lagrangian is a function of Φ and the covariant derivative $D\Phi$ is with respect to a complex vertical connection, particularly for simplicity the Bott connection. Indeed, L_p depends implicitly by the base point $u = (z,\eta) \in E$. Then by complex Legendre transformation (2.18), (2.19), the sections of VE (plus their conjugates) will be send into sections of VE^*. We obtain hereby the field particles on E^*:

$$\Phi_a(z,\zeta) = h_{a\bar{b}}\Phi^{\bar{b}}(z,\eta := \frac{\partial H_p}{\partial\zeta}) = \left(\frac{\partial L_p}{\partial\Phi^a}\right)^*. \tag{4.1}$$

Consequently, by (2.19) we obtain a Hamiltonian for the $\mathcal{L}$- dual particle $\Phi^* = \Phi_a\dot{\partial}^a$,

$$H_p(\Phi^*) = \Phi_a\Phi^a + \bar{\Phi}_a\bar{\Phi}^a - L_p(\Phi). \tag{4.2}$$

We note that H_p is gauge invariant with respect to the $\mathcal{L}$-dual gauge transformation $\overset{*}{\Upsilon}$ of Υ, forasmuch L_p is gauge with respect to Υ. As well, we proved that the $\mathcal{L}$-dual of a vertical connection on VE is a vertical connection VE^*, i.e. $(D)^* = \overset{*}{D}$, and moreover if one is gauge the other is gauge too. Hence, $L_p(\Phi^a, D\Phi^a)$ by (4.2) determines the $\mathcal{L}$-dual Hamiltonian $H_p(\Phi_a, \overset{*}{D}\,\Phi_a)$.

Now, by taking $\overset{*}{D}\,\Phi_a$ as an independent variable for the Hamiltonian, we can write down the following variation

$$\delta H = \frac{\partial H}{\partial\Phi_a}(\delta\Phi_a) + \frac{\partial H}{\partial\Phi_{a\,|\,i}}(\delta\Phi_{a\,|\,i}) + \frac{\partial H}{\partial\Phi_{a\,|\,^b}}(\delta\Phi_{a\,|\,^b}) + \ conjugates$$

By the same symbols ω and θ from (2.20) we denoting the $\mathcal{L}$-dual simplectic forms associated to the variations of field particle. Thus, we may write θ as being

$$\theta = h^{\bar{b}a}\{\delta\Phi_a \wedge \delta\bar{\Phi}_b + \sum_i \delta\Phi_{a\,|\,i} \wedge \delta\bar{\Phi}_{b\,|\,i} + \sum_c \delta\Phi_{a\,|\,^c} \wedge \delta\bar{\Phi}_{b\,|\,^c}\} \tag{4.3}$$

Let as associate to Φ^a, on the curve $t \to \Phi^a(z(t), \eta(t))$, the vector field X_{Φ^a}

$$X_{\Phi^a} = \frac{\delta\Phi^a}{dt}\frac{\delta}{\delta\Phi^a} + \sum_i \frac{\delta\Phi^a_{\,|\,i}}{dt}\frac{\delta}{\delta\Phi^a_{\,|\,i}} + \sum_b \frac{\delta\Phi^a_{\,|\,b}}{dt}\frac{\delta}{\delta\Phi^a_{\,|\,b}} + conjugates.$$

By $\mathcal{L}$-duality on the curve $t \to \Phi_a(z(t), \zeta(t))$ we obtain the vector field $\overset{*}{X}_{\Phi_a} = h^{\bar{b}a}\,(X_{\bar{\Phi}^b})^*$,

$$\overset{*}{X}_{\Phi^a} = \frac{\delta\Phi_a}{dt}\frac{\delta}{\delta\Phi_a} + \sum_i \frac{\delta\Phi_{a\,|\,i}}{dt}\frac{\delta}{\delta\Phi_{a\,|\,i}} + \sum_b \frac{\delta\Phi_{a\,|\,^b}}{dt}\frac{\delta}{\delta\Phi_{a\,|\,^b}} + conjugates.$$

The requirement $i_{\overset{*}{X}_{\Phi^a}}\theta = \delta H$ of integral curve for $\overset{*}{X}_{\Phi^a}$ yields

$$h^{\bar{b}a}\frac{\delta\bar{\Phi}_b}{dt} = -\frac{\partial H}{\partial\Phi_a}\;;\quad h^{\bar{b}a}\frac{\delta\bar{\Phi}_{b\,|\,i}}{dt} = -\frac{\partial H}{\partial\Phi_{a\,|\,i}}\;;\quad h^{\bar{b}a}\frac{\delta\bar{\Phi}_{b\,|\,^c}}{dt} = -\frac{\partial H}{\partial\Phi_{a\,|\,^c}}.$$

Tacking variations $\delta\Phi_a$ in (2.17), we easily can check that $(\delta\Phi_a)_{\,|\,i} = \delta(\Phi_{a\,|\,i})$ and $(\delta\Phi_a)_{\,|\,^c} = \delta(\Phi_{a\,|\,^c})$ and hence, from the above formulas is obtain

$$h^{\bar{b}a}\frac{\delta\bar{\Phi}_b}{dt} = -\frac{\partial H}{\partial\Phi_a}\;;\quad \left(\frac{\partial H}{\partial\Phi_a}\right)_{\,|\,i} = \frac{\partial H}{\partial(\Phi_{a\,|\,i})}\;;\quad \left(\frac{\partial H}{\partial\Phi_a}\right)_{\,|\,^c} = \frac{\partial H}{\partial(\Phi_{a\,|\,^c})} \tag{4.4}$$

called the *complex Hamilton field equations.*

By $\mathcal{L}$-duality let us obtain now from (2.5) the Klein-Gordon type Hamiltonian. Since $\gamma_{i\bar{j}}(z)$ is a Hermitian metric on the base manifold M, we identify it with $(\gamma_{i\bar{j}}(z))^*$ on E^*. For the Hermitian mass metric $\gamma_{a\bar{b}}(z,\eta)$, (or eventually for one which comes from an external Lagrangian of Antonelli-Shimada type, for instance), we recall from [Mu3] that the $\mathcal{L}-$dual of a complex Lagrange (Finsler) space is a complex Hamilton (Cartan) space and their metrics correspond by $\mathcal{L}-$duality. So, let us setting $\tau_{a\bar{b}} := (\gamma_{a\bar{b}})^*$ and then $\tau^{\bar{b}a}$ its inverse. Then the associated Klein-Gordon Hamiltonian to Φ_a particle is

$$H_p(\Phi^*, \overset{*}{D}\,\Phi^*) = -\frac{1}{2}\sum_a\{\gamma^{\bar{j}i}\,\overset{*}{D}_{\delta^*_i}\,\Phi_a\,\overset{*}{D}_{\delta^*_{\bar{j}}}\,\bar{\Phi}_a + \tau^{\bar{b}c}\,\overset{*}{D}_{\dot{\partial}^c}\,\Phi_a D_{\dot{\partial}^b}\bar{\Phi}_a\} - (V(\Phi))^*. \tag{4.5}$$

Because its metric tensor is the $\mathcal{L}$-dual of the Lagrangian particle metric tensor, $h_{a\bar{b}} = (g_{a\bar{b}})^*$, the corresponding Hamiltonian density to the Lagrangian density $\mathcal{L}_p = L_p \mid g \mid^2$ will be $\mathcal{H}_p = H_p \mid g \mid^{-2}$.

For the Yang-Mills Hamiltonian we take into account the Proposition 2.6 and Proposition 2.7 and therefore we obtain a complex Hamilonian which contains only the curvature of a vertical connection on E^*. Although the Bott complex connections don't correspond by $\mathcal{L}$-duality, for applications is useful the following Y-M Hamiltonian,

$$H_{YM}(\overset{*B}{D}) = \frac{1}{2}\sum_{a,b}\gamma^{\bar{j}i}\gamma^{\bar{k}l}\,\overset{*}{R}{}^a_{bi\bar{j}}\overset{*}{R}{}^a_{bl\bar{k}}\;. \tag{4.6}$$

Finally, if we consider for (4.5) its metric tensor $h_{a\bar{b}} = (g_{a\bar{b}})^*$, we may constuct the Ricci curvatures for γ and h on VE^* and thereafter the Ricci scalars, $\overset{*}{\rho}\,(\gamma)$ and $\overset{*}{\rho}\,(h)$. We observe that $\overset{*}{\rho}\,(\gamma)$ is identified with $\rho(\gamma)$. Thus, the Hilbert-Einstein gravitational Hamiltonian is $H_{HE} = \frac{1}{\chi}\,\overset{*}{\rho}$, where $\overset{*}{\rho} = \rho(\gamma) + \overset{*}{\rho}\,(h)$.

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