ON THE WORLD FUNCTION AND THE RELATION BETWEEN GEOMETRIES

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It is shown that the World function can be regarded as a link between the qualitatively different geometries with one and the same congruence of the world lines (geodesics). If the space in which the World function is defined is a polynumber space, then the hypothesis of the analyticity of the vector field of the generalized velocities of the world lines lead to the strict limitations on the structure of the World function. Main result: Minkowskian space and polynumber space correspond to the same physical World.

1 Introduction

The idea that everything happening in the physical world is governed by a single scalar function has originated long ago and can hardly be attributed to any scientist or even to a group of scientists. It is this function that will be called the World function here. For example, H. Weyl [1] uses the term "World function" when discussing the Mie theory. It is not definitely clear what to choose as a World function. G. Mie in his theory (a field theory) suggested to choose the Lagrangian of the field, i.e. to take the density of the Lagrange function as a World function. In this paper the field equations and the field theories will not be discussed.

For the observer using the classical mechanics and Finsler geometry, it is sufficient to know how all the material points move, in other words it is sufficient to know the congruence of the world lines in the space-time. In Finsler geometry such a congruence is a normal congruence of the geodesics [2], i.e. there exists such a scalar function, S, the level surfaces of which are transversal to the given congruence of geodesics. In classical mechanics such function is usually called 'action as a function of coordinates'. In this paper it is this finction, S, that will be considered the World function.

So, let us adopt that in the coordinate space $x^1, x^2, ..., x^n$ the scalar function $S(x)$ corresponding to the notion of *action as a fuction of coordinates* $x^1, x^2, ..., x^n$ in classical mechanics plays a role of the World function. Taken as it is, the scalar function, S, can not define the field of velocities and, therefore, can not define a congruence of geodesics each of which corresponds to an observer or to a material particle. One needs an additional procedure, $\tilde{\varphi}$, providing the possibility to pass from the covariant 'vectors' to the contravariant 'vectors'. In any Finsler geometry, Φ_n, there is such a procedure. Thus, the pair $\{S; \tilde{\varphi}\}$ as well as the pair $\{S; \Phi_n\}$ defines the congruence of the world lines for all the points of the space, that is defines the evolution of this space.

Let x^0, x^1, x^2, x^3 be the Minkowskian space with the length element defined as

$$ds = mc\sqrt{(x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2} \equiv mc\sqrt{\overset{o}{g}_{ij}\; x^i x^j}\,, \qquad (1)$$

where the factor mc – optional from the geometric point of view – provides a better possibility to give a physical interpretation of the geometrical objects; m and c are the rest mass of the particle and the light velocity in vacuum. The tangent equation of the indicatrix in such a space can be written as follows

$$(p_0)^2 - (p_1)^2 - (p_2)^2 - (p_3)^2 = (mc)^2 \,. \tag{2}$$

Then, $S(x^0, x^1, x^2, x^3)$, the action as a function of coordinates in the Minkowski space must suffice the Hamilton-Jacoby equation

$$\left(\frac{dS}{dx^0}\right)^2 - \left(\frac{dS}{dx^1}\right)^2 - \left(\frac{dS}{dx^2}\right)^2 - \left(\frac{dS}{dx^3}\right)^2 = (mc)^2 \,. \tag{3}$$

Let us now take an arbitrary function $\tilde{S}$ which suffice

$$\left(\frac{d\tilde{S}}{dx^0}\right)^2 - \left(\frac{d\tilde{S}}{dx^1}\right)^2 - \left(\frac{d\tilde{S}}{dx^2}\right)^2 - \left(\frac{d\tilde{S}}{dx^3}\right)^2 > 0 \,. \tag{4}$$

and substitute it into (3). The result is that the function $\tilde{S}$ is a solution of the Hamilton-Jacoby equation which corresponds to the Finsler geometry with the length element

$$d\tilde{s} = \kappa(x) \cdot mc\sqrt{(x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2} \tag{5}$$

and the tangent equation of the indicatrix

$$(p_0)^2 - (p_1)^2 - (p_2)^2 - (p_3)^2 = \kappa(x)^2 \cdot (mc)^2 \,, \tag{6}$$

where

$$\kappa(x) \equiv \frac{1}{mc}\sqrt{\left(\frac{d\tilde{S}}{dx^0}\right)^2 - \left(\frac{d\tilde{S}}{dx^1}\right)^2 - \left(\frac{d\tilde{S}}{dx^2}\right)^2 - \left(\frac{d\tilde{S}}{dx^3}\right)^2} \,. \tag{7}$$

Notice, that if the length elements of two geometries, ds, $d\tilde{s}$, defined in the same coordinate space $x^1, x^2, ..., x^n$ are related as

$$d\tilde{s} = \kappa(x) ds \,, \tag{8}$$

where $k(x) > 0$ is an arbitrary function of a point, then these two geometries are called conformly connected [2]. The geometry $d\tilde{s}$ differs from the geometry ds in such a way that in every infinitely small vicinity of every point of space, $x^1, x^2, ..., x^n$, there is a scale transformation while the extension-contraction coefficient, $\kappa(x)$, depends on the point.

Thus, we see that if we know the arbitrary scalar function, $\tilde{S}$, sufficing (4) in the flat Minkowski space (1), then we know the World function in the space given by (5) which is conformly connected to the Minkowski space. Therefore, the world lines equations in space (5) can be written as:

$$\dot{x}^i = \overset{o}{g}{}^{ij} \frac{d\tilde{S}}{dx^j} \lambda(x, y) \,, \tag{9}$$

where $\dot{x}^i \equiv \frac{dx^i}{d\tau}$ is a derivative over τ (evolution parameter) along the world line, and $\lambda(x, y) > 0$ is an arbitrary function.

All the above said is true (with regard to the obvious changes of notation in formulas) for the Euclidean or for pseudo Euclidean geometry of the arbitrary dimension n, but only for $n = 2$ one could correlate a system of the associative commutative non-degenerate numbers (correspondingly, complex numbers, C_2, and hyperbolic numbers, H_2), to the Euclidean or to pseudo Euclidean space.

In this approach the form of the World function is not limited by anything but (4). To make the form of the World function concrete for the polynumber space, P_n, one could use the analyticity condition - the condition giving a relation between the World function and the analytical functions of the polynumber variable, P_n. In this paper this is done in the form of Hypotheses I, II. The other realizations are also possible.

1.1 Complex plane

Hypothesis I_{C_2}: Components of the vector field that produces the world lines corresponding to the given World function, are the components of the analytical function of the complex variable.

According to this Hypothesis

$$\lambda(x,y)\cdot\frac{\partial\tilde{S}}{\partial x}=u\,,\qquad \lambda(x,y)\cdot\frac{\partial\tilde{S}}{\partial y}=v\,, \tag{10}$$

where $F(z)=u(x,y)+iv(x,y)$ is an analytical function of the complex variable $z=x+iy$. Then the Cauchi-Riemann relations give the following partial differential equations for the World function $\tilde{S}$:

$$\frac{\partial}{\partial x}\lambda(x,y)\frac{\partial\tilde{S}}{\partial x}=\frac{\partial}{\partial y}\lambda(x,y)\frac{\partial\tilde{S}}{\partial y}\,,\qquad \frac{\partial}{\partial y}\lambda(x,y)\frac{\partial\tilde{S}}{\partial x}=-\frac{\partial}{\partial x}\lambda(x,y)\frac{\partial\tilde{S}}{\partial y}\,. \tag{11}$$

If $\lambda(x,y)\equiv 1$, then the equations (11) simplify:

$$\frac{\partial^2\tilde{S}}{\partial x^2}-\frac{\partial^2\tilde{S}}{\partial y^2}=0\,,\qquad \frac{\partial^2\tilde{S}}{\partial x\partial y}=0\,. \tag{12}$$

The general solution of this system of equations is

$$\tilde{S}=\frac{A}{2}(x^2+y^2)+a_1x+a_2y+b\,, \tag{13}$$

where A,a_1,a_2,b are real numbers. Notice, that if $A\neq 0$, then function $\tilde{S}$ is not a component of the analytical function of complex variable.

Hypothesis II_{C_2}: The components of the vector field that produces the world lines corresponding to the given World function, are the components of the function of complex variable conjugate to the analytical function of complex variable.

Then according to this hypothesis

$$\lambda(x,y)\cdot\frac{\partial\tilde{S}}{\partial x}=u\,,\qquad \lambda(x,y)\cdot\frac{\partial\tilde{S}}{\partial y}=-v\,, \tag{14}$$

where $F(z)=u(x,y)+iv(x,y)$ is an analytical function of complex variable $z=x+iy$. The Cauchi-Riemann relations give the following partial differential equations for the World function $\tilde{S}$:

$$\frac{\partial}{\partial x}\lambda(x,y)\frac{\partial\tilde{S}}{\partial x}=-\frac{\partial}{\partial y}\lambda(x,y)\frac{\partial\tilde{S}}{\partial y}\,,\qquad \frac{\partial}{\partial y}\lambda(x,y)\frac{\partial\tilde{S}}{\partial x}=\frac{\partial}{\partial x}\lambda(x,y)\frac{\partial\tilde{S}}{\partial y}\,. \tag{15}$$

If $\lambda(x,y)\equiv 1$, then the equations (15) simplify and give a single partial differential equation:

$$\frac{\partial^2\tilde{S}}{\partial x^2}+\frac{\partial^2\tilde{S}}{\partial y^2}=0\,. \tag{16}$$

Thus, provided the Hypothesis II_{C_2} is true and $\lambda(x,y)\equiv 1$, the function $\tilde{S}$ is a component of the analytical function of the complex variable, and the corresponding geometry which is conformly connected to the Euclidean plane can be obtained with the help of the conformal transformation of the Euclidean plane.

1.2 Hyperbolic plane

The metric tensor for the hyperbolic plane has the form

$$\overset{o}{g}_{ij}= diag(1,-1)\,, \tag{17}$$

and the Cauchi-Riemann relations for the analytical functions $F(z) = u(x,y) + jv(x,y)$ of the variable $H_2 \ni z = x + jy$, $j^2 = 1$ can be written as:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}\,, \qquad \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}\,. \tag{18}$$

Hypothesis I_{H_2}: Components of the vector field that produces the world lines corresponding to the given World function, are the components of the analytical function of the variable H_2.

According to this hypothesis and in analogy to (9) for $n = 2$, one gets

$$\lambda(x,y)\cdot\frac{\partial\tilde S}{\partial x} = u\,, \qquad \lambda(x,y)\cdot\frac{\partial\tilde S}{\partial y} = -v\,, \tag{19}$$

where $F(z) = u(x,y) + iv(x,y)$ is an analytical function of the variable $H_2 \ni z = x + jy$. Then the Cauchi-Riemann relations give the following partial differential equations for the World function, $\tilde S$:

$$\frac{\partial}{\partial x}\lambda(x,y)\frac{\partial\tilde S}{\partial x} = -\frac{\partial}{\partial y}\lambda(x,y)\frac{\partial\tilde S}{\partial y}\,, \qquad \frac{\partial}{\partial y}\lambda(x,y)\frac{\partial\tilde S}{\partial x} = -\frac{\partial}{\partial x}\lambda(x,y)\frac{\partial\tilde S}{\partial y}\,. \tag{20}$$

If $\lambda(x,y) \equiv 1$, then the equations (20) simplify:

$$\frac{\partial^2\tilde S}{\partial x^2} + \frac{\partial^2\tilde S}{\partial y^2} = 0\,, \qquad \frac{\partial^2\tilde S}{\partial x\partial y} = 0\,. \tag{21}$$

The general solution of this system of equations is

$$\tilde S = \frac{A}{2}(x^2 - y^2) + a_1x + a_2y + b\,, \tag{22}$$

where A, a_1, a_2, b are real numbers. Notice, that if $A \neq 0$, function $\tilde S$ is not a component of the analytical function of variable H_2.

Hypothesis II_{H_2}: The components of the vector field that produces the world lines corresponding to the given World function, are the components of the function of variable H_2 conjugate to the analytical function of the variable H_2.

According to this hypothesis

$$\lambda(x,y)\cdot\frac{\partial\tilde S}{\partial x} = u\,, \qquad \lambda(x,y)\cdot\frac{\partial\tilde S}{\partial y} = v\,, \tag{23}$$

where $F(z) = u(x,y) + jv(x,y)$ is an analytical function of the variable $H_2 \ni z = x + iy$. Then the Cauchi-Riemann relations give the following partial differential equations for the World function $\tilde S$:

$$\frac{\partial}{\partial x}\lambda(x,y)\frac{\partial\tilde S}{\partial x} = \frac{\partial}{\partial y}\lambda(x,y)\frac{\partial\tilde S}{\partial y}\,, \qquad \frac{\partial}{\partial y}\lambda(x,y)\frac{\partial\tilde S}{\partial x} = \frac{\partial}{\partial x}\lambda(x,y)\frac{\partial\tilde S}{\partial y}\,. \tag{24}$$

If $\lambda(x, y) \equiv 1$, then the equations (24) simplify and give a single partial differential equation:

$$\frac{\partial^2 \tilde{S}}{\partial x^2} - \frac{\partial^2 \tilde{S}}{\partial y^2} = 0\,. \tag{25}$$

Thus, provided the Hypothesis II_{H_2} is true and $\lambda(x, y) \equiv 1$, the function $\tilde{S}$ is a component of the analytical function of the variable H_2, and the corresponding geometry which is conformly connected to the hyperbolic plane can be obtained with the help of the conformal transformation of the hyperbolic plane.

2 Polynumbers P_n

Let us regard a system of the non-degenerate n-numbers P_n. The corresponding coordinate space, $x^1, x^2, ..., x^n$, is a Finsler metric space with the length element

$$ds = mc\sqrt[n]{\overset{o}{g}_{i_1 i_2 ... i_n}\; dx^{i_1} dx^{i_2} ... dx^{i_n}}\,, \tag{26}$$

$\overset{o}{g}_{i_1 i_2 ... i_n}$ is a metric tensor that does not depend on point. The Finsler spaces of this kind have been studied in mathematical literature for a long time (see, for example, [3] - [6]), but the fact that all the polynumber spaces are just the Finsler spaces of this type was discovered not long ago in [7], [8] and the subsequent papers of the same authors.

The components of the generalized momentum in the geometry (26) can be calculated according to the formulas:

$$p_i = mc\frac{\overset{o}{g}_{i j_2 ... j_n}\; dx^{j_2} ... dx^{j_n}}{\left(\overset{o}{g}_{i_1 i_2 ... i_n}\; dx^{i_1} dx^{i_2} ... dx^{i_n}\right)^{\frac{n-1}{n}}}\,. \tag{27}$$

Finsler geometry with the length element (26) will be called resolvable if the tangent equation for the indicatrix can be written as

$$\overset{o}{g}{}^{i_1 i_2 ... i_n}\; p_{i_1} p_{i_2} ... p_{i_n} = \mu^n (mc)^n\,, \tag{28}$$

where $\mu > 0$ is a constant. For the Riemannian or pseudo Riemannian geometry the re solvability means that the determinant of the metric tensor is not equal to zero. It seems that the Finsler geometry in the space of the non-degenerate polynumbers is always resolvable, but this statement demands the strict proof.

As it can be seen from expressions (26) - (28), tensors $\overset{o}{g}_{i_1 i_2 ... i_n}$, $\overset{o}{g}{}^{i_1 i_2 ... i_n}$ must suffice the following relation of the resolvable Finsler geometry

$$\begin{aligned} &\overset{o}{g}{}^{j_1 j_2 ... j_n} \times \\ &\times \overset{o}{g}_{j_1 i_2 ... i_n}\; dx^{i_2} ... dx^{i_n}\; \overset{o}{g}_{j_2 k_2 ... k_n}\; dx^{k_2} ... dx^{k_n} ... \overset{o}{g}_{j_n m_2 ... m_n}\; dx^{m_2} ... dx^{m_n} = \\ &= \mu^n \left(\overset{o}{g}_{i_1 i_2 ... i_n}\; dx^{i_1} dx^{i_2} ... dx^{i_n}\right)^{n-1}\,. \end{aligned} \tag{29}$$

Action as a function of coordinates in geometry (26) suffices the Hamilton-Jacoby equation:

$$\overset{o}{g}{}^{j_1 j_2 ... j_n}\frac{\partial \tilde{S}}{\partial x^{j_1}}\frac{\partial \tilde{S}}{\partial x^{j_2}} ... \frac{\partial \tilde{S}}{\partial x^{j_n}} = \mu^n (mc)^n\,. \tag{30}$$

Let us regard an arbitrary World function, $\tilde{S}(x^1, x^2, ..., x^n)$, with the only condition

$$\overset{o}{g}{}^{j_1 j_2 \dots j_n} \frac{\partial \tilde{S}}{\partial x^{j_1}} \frac{\partial \tilde{S}}{\partial x^{j_2}} \dots \frac{\partial \tilde{S}}{\partial x^{j_n}} > 0 \,, \tag{31}$$

Then function $\tilde{S}(x)$ is the action for the geometry conformly connected to geometry (26) with the length element

$$d\tilde{s} = \kappa(x) \cdot mc \sqrt[n]{\overset{o}{g}_{i_1 i_2 \dots i_n} \, x^{i_1} x^{i_2} \dots x^{i_n}} \,, \tag{32}$$

where $\kappa(x) > 0$ is the extension–contraction coefficient which varies from point to point of the coordinate space

$$\kappa(x) = \frac{1}{\mu \cdot mc} \sqrt[n]{\overset{o}{g}{}^{j_1 j_2 \dots j_n} \frac{\partial \tilde{S}}{\partial x^{j_1}} \frac{\partial \tilde{S}}{\partial x^{j_2}} \dots \frac{\partial \tilde{S}}{\partial x^{j_n}}} \,, \tag{33}$$

and the World function, $\tilde{S}$, is the solution of the Hamilton-Jacoby equation of the following form:

$$\overset{o}{g}{}^{j_1 j_2 \dots j_n} \frac{\partial \tilde{S}}{\partial x^{j_1}} \frac{\partial \tilde{S}}{\partial x^{j_2}} \dots \frac{\partial \tilde{S}}{\partial x^{j_n}} = \kappa(x)^n \cdot \mu^n (mc)^n \,. \tag{34}$$

The field of velocities that defines the congruence of the world lines can be expressed by the World function, $\tilde{S}$, by the formula

$$\dot{x}^i = \overset{o}{g}{}^{i j_2 \dots j_n} \frac{\partial \tilde{S}}{\partial x^{j_2}} \dots \frac{\partial \tilde{S}}{\partial x^{j_n}} \cdot \lambda(x)^{n-1} \,, \tag{35}$$

where $\lambda(x) > 0$ is an arbitrary scalar function.

The algebra of polynumbers $P_n \ni X = x^1 e_1 + x^2 e_2 + ... + x^n e_n$ is completely defined by the multiplication rule for the basis elements:

$$e_i e_j = p^k_{ij} e_k \tag{36}$$

that is by the number tensor, p^k_{ij}. Notice, that the polynumbers, P_n, are called non-degenerate if

$$det(q_{ij}) \neq 0 \,, \qquad q_{ij} \equiv p^k_{im} p^m_{kj} \,. \tag{37}$$

In this case one can construct tensor q^{ij}. If ϵ^i are the coefficients of the expansion of the unity $1 \in P_n$ in the basis e_i, then the Cauchi-Riemann relation for the analytical function $F(X) = f(x)^i e_i$ of the variable P_n can be written in the following form:

$$\frac{\partial f^i}{\partial x^k} - p^k_{ij} \epsilon^m \frac{\partial f^j}{\partial x^m} = 0 \,. \tag{38}$$

Hypothesis I_{P_n}: Components of the vector field that produces the world lines corresponding to the given World function, are the components of the analytical function of the variable P_n.

If $F(X) = f(x)^i e_i$ is an analytical function of the variable P_n, then this hypothesis leads to the expression:

$$f^i(x^1, x^2, ..., x^n) = \overset{o}{g}{}^{i j_2 \dots j_n} \frac{\partial \tilde{S}}{\partial x^{j_2}} \dots \frac{\partial \tilde{S}}{\partial x^{j_n}} \cdot \lambda(x)^{n-1} \,. \tag{39}$$

Substituting these components of the analytical function expressed by the World function into the Cauchi-Riemann relations, we get such a system of partial differential equations that if Hypothesis I_{P_n} is fulfilled, then the World function suffices this system.

Hypothesis II_{P_2}: Components of the vector field that produces the world lines corresponding to the given World function, are the components of the function of the variable P_n conjugate to the analytical function of the same variable with the help of a special unary operation.

Let us define the unary operation $\bar{X} = Y$ acting on the set $P_n \ni X, Y$ in the following way:

$$y^i = \overset{o}{g}{}^{ij_2...j_n} q_{j_2m_2}...q_{j_nm_n} x^{m_2}...x^{m_n} . \qquad (40)$$

For complex numbers C_2 and hyperbolic numbers H_2 such a unary operation is a regular conjugation, while on the polynumber set H_4 (and H_n) this operation coincides with the operation of normal conjugation [9] within the accuracy of a number factor. The unary operation (40) can be generalized for $(n-1)$ arguments, it will remain symmetrical due to its definition. To distinguish such a unary operation and a corresponding $(n-1)$-ary operation from other conjugations in the polynumber algebras, let us call such an operation *symmetrical conjugation*.

Comparing formulas (35) and (40) and changing x^i to f^i, we see that the realization of the *Hypothesis II_{P_2}* leads to the relations

$$q_{ij} f^j = \frac{\partial \tilde{S}}{\partial x^i} \lambda(x) , \qquad (41)$$

or

$$f^i = q^{ij} \frac{\partial \tilde{S}}{\partial x^j} \lambda(x) , \qquad (42)$$

that is the quantities $q^{ij} \frac{\partial \tilde{S}}{\partial x^j} \lambda(x)$ are the components of the analytical function of the variable P_n.

Let us show that one and the same pair {World function; congruence of the world lines} can be realized in various Finsler geometries.

We introduce the notation

$$g^{ij}(x) = \left[\frac{1}{\kappa(x) \cdot \mu \cdot cm}\right]^{n-2} \overset{o}{g}{}^{ijj_3...j_n} \frac{\partial \tilde{S}}{\partial x^{j_3}} ... \frac{\partial \tilde{S}}{\partial x^{j_n}} . \qquad (43)$$

Let $det(g^{ij}(x)) \neq 0$, then we can construct the twice covariant tensor $g_{ij}(x)$. Let us regard the pseudo Riemannian geometry with the length element

$$ds' = \kappa(x) \cdot \mu \cdot cm \sqrt{g_{ij} dx^i dx^j} . \qquad (44)$$

The tangent equation of the indicatrix in such a geometry is

$$g^{ij} p_i p_j = \kappa(x)^2 \cdot \mu^2 \cdot (cm)^2 , \qquad (45)$$

and the Hamilton-Jacoby equation for the action $S'(x)$ is

$$g^{ij} \frac{\partial S'}{\partial x^i} \frac{\partial S'}{\partial x^j} = \kappa(x)^2 \cdot \mu^2 \cdot (cm)^2 . \qquad (46)$$

Substitute the expression (43) into this equation and get

$$\overset{o}{g}{}^{j_1j_2j_3...j_n} \frac{\partial S'}{\partial x^{j_1}} \frac{\partial S'}{\partial x^{j_2}} \frac{\partial \tilde{S}}{\partial x^{j_3}} ... \frac{\partial \tilde{S}}{\partial x^{j_n}} = \kappa(x)^n \cdot \mu^n (mc)^n . \qquad (47)$$

Thus, we see that the function $S' = \tilde{S}$ is the solution of the equation (46), that is function $\tilde{S}$ remains the World function in the geometry (44).

The field of velocities in the geometry (44) is defined by the formula

$$\dot{x}^i = g^{ij} \frac{\partial \tilde{S}}{\partial x^j} \cdot \lambda'(x) \,, \tag{48}$$

where $\lambda'(x) > 0$ is a scalar function. Substitute the expression (43) into this equation

$$\lambda'(x) = \kappa(x)^{n-2} \cdot \mu^{n-2} \cdot (cm)^{n-2} \cdot \lambda(x)^{n-1} \tag{49}$$

and get the formula

$$\dot{x}^i = \overset{o}{g}{}^{ij_2...j_n} \frac{\partial \tilde{S}}{\partial x^{j_2}} ... \frac{\partial \tilde{S}}{\partial x^{j_n}} \cdot \lambda(x)^{n-1}, \tag{50}$$

which coincides with the formula (35).

So, one and the same pair {World function; congruence of the world lines} can be realized in the qualitatively different geometries.

One can use metric tensor $g_{i_1 i_2 ... i_m}(x^{i_1} x^{i_2} ... x^{i_n})$ to obtain metric tensor with less number of indices, $r < m$. To do this one should contract some indices with vector or tensor contravariant fields (see, for example, [3] - [6]). The speculations given above show that the best method of contraction for polynumber spaces P_n is the following: $g_{i_1 i_2 ... i_r}(x^{i_1} x^{i_2} ... x^{i_n}) = a(x) \cdot g_{i_1 i_2 ... i_m}(x^{i_1} x^{i_2} ... x^{i_n}) f^{i_{r+1}}_{(1)} f^{i_{r+2}}_{(2)} ... f^{i_m}_{(m-r)}$, where $a(x)$ is some scalar function and $f^i_{(A)}$ are the components of the analytical functions of variable P_n or the components of some conjugated functions to the analytical functions of the same variable.

3 Hypercomplex numbers H_4

Notice, that the system of hypercomplex numbers H_4 is isomorphic to the algebra of real square diagonal matrices 4×4. The corresponding coordinate space is the metric Finsler space with Berwald-Moor metric. In H_4 there is a special basis, e_1, e_2, e_3, e_4, with the following multiplication rule

$$e_i e_j = p^k_{ij} e_k \,, \qquad p^k_{ij} = \begin{cases} 1\,, & i = j = k\,, \\ 0\,, & \end{cases} \tag{51}$$

The components of tensors q_{ij} (37), q^{ij} in this basis give a unity matrix:

$$(q_{ij}) = (q_{ij}) = diag(1, 1, 1, 1)\,. \tag{52}$$

The length element in the H_4 space in the special basis (51) is

$$ds = mc\sqrt[4]{dx^1 dx^2 dx^3 dx^4} \equiv mc\sqrt[4]{\overset{o}{g}_{ijkm}\, dx^i dx^j dx^k dx^m}\,, \tag{53}$$

where

$$\overset{o}{g}_{ijkm} = \begin{cases} \frac{1}{24}\,, & for\, alldifferent\; i, j, k, m \\ 0\,, & else\;\; . \end{cases} \tag{54}$$

The components of the generalized momentum are defined by the formula

$$p_i = \frac{mc}{4} \cdot \frac{\sqrt[4]{dx^1 dx^2 dx^3 dx^4}}{dx^i}\,, \tag{55}$$

and the tangent equation of the indicatrix is

$$p_1 p_2 p_3 p_4 = \left(\frac{mc}{4}\right)^4 , \tag{56}$$

or in the covariant form

$$\overset{o}{g}{}^{ijkm} \; p_i p_j p_k p_m = \left(\frac{mc}{4}\right)^4 , \tag{57}$$

For the special basis used above, we have

$$\left(\overset{o}{g}{}^{ijkm}\right) = \left(\overset{o}{g}_{ijkm}\right) . \tag{58}$$

Action as a function of coordinates in the H_4 space suffices the equation

$$\overset{o}{g}{}^{ijkm} \frac{\partial \tilde{S}}{\partial x^i} \frac{\partial \tilde{S}}{\partial x^j} \frac{\partial \tilde{S}}{\partial x^k} \frac{\partial \tilde{S}}{\partial x^m} = \left(\frac{mc}{4}\right)^4 , \tag{59}$$

or

$$\frac{\partial \tilde{S}}{\partial x^1} \frac{\partial \tilde{S}}{\partial x^2} \frac{\partial \tilde{S}}{\partial x^3} \frac{\partial \tilde{S}}{\partial x^4} = \left(\frac{mc}{4}\right)^4 . \tag{60}$$

Substitute into (60) some World function, $\tilde{S}(x)$, that suffices the only condition

$$\frac{\partial \tilde{S}}{\partial x^1} \frac{\partial \tilde{S}}{\partial x^2} \frac{\partial \tilde{S}}{\partial x^3} \frac{\partial \tilde{S}}{\partial x^4} > 0 , \tag{61}$$

and get

$$\frac{\partial \tilde{S}}{\partial x^1} \frac{\partial \tilde{S}}{\partial x^2} \frac{\partial \tilde{S}}{\partial x^3} \frac{\partial \tilde{S}}{\partial x^4} = \kappa(x)^4 \cdot \left(\frac{mc}{4}\right)^4 , \tag{62}$$

This means that the function, $\tilde{S}(x)$, is a World function in the geometry which is conformly connected to the Berwald-Moor geometry (53), which is a geometry with the length element

$$ds = \kappa(x) \cdot mc \sqrt[4]{dx^1 dx^2 dx^3 dx^4} , \tag{63}$$

The extension-contraction coefficient is given by

$$\kappa(x) = \frac{4}{mc} \sqrt[4]{\frac{\partial \tilde{S}}{\partial x^1} \frac{\partial \tilde{S}}{\partial x^2} \frac{\partial \tilde{S}}{\partial x^3} \frac{\partial \tilde{S}}{\partial x^4}} . \tag{64}$$

In this geometry the field of velocities defining the congruence of the world lines is

$$\dot{x}^i = \frac{\frac{\partial \tilde{S}}{\partial x^1} \frac{\partial \tilde{S}}{\partial x^2} \frac{\partial \tilde{S}}{\partial x^3} \frac{\partial \tilde{S}}{\partial x^4}}{\frac{\partial \tilde{S}}{\partial x^i}} \cdot \lambda(x)^3 , \tag{65}$$

where $\lambda(x) > 0$ is a scalar function.

Hypothesis I_{H_4}: Components of the vector field that produces the world lines corresponding to the given World function, are the components of the analytical function of the variable H_4.

In the special basis in question an arbitrary analytical function of the variable H_4 has the form

$$F(X) = f^1(x^1)e_1 + f^2(x^2)e_2 + f^3(x^3)e_3 + f^4(x^4)e_4 , \tag{66}$$

where f^i are the arbitrary functions of a single real variable. That is why the Hypothesis I_{H_4} leads to the demand

$$f^i(x^{i-}) = \frac{\frac{\partial\tilde{S}}{\partial x^1}\frac{\partial\tilde{S}}{\partial x^2}\frac{\partial\tilde{S}}{\partial x^3}\frac{\partial\tilde{S}}{\partial x^4}}{\frac{\partial\tilde{S}}{\partial x^i}} \cdot \lambda(x)^3 \,. \tag{67}$$

Multipliing the expressions (67) with different indices and performing some transformations, one gets

$$\frac{\partial\tilde{S}}{\partial x^1}\frac{\partial\tilde{S}}{\partial x^2}\frac{\partial\tilde{S}}{\partial x^3}\frac{\partial\tilde{S}}{\partial x^4} = \frac{\sqrt[3]{f^1 f^2 f^3 f^4}}{\lambda^4} \tag{68}$$

and

$$\frac{\partial\tilde{S}}{\partial x^i} = \frac{\sqrt[3]{f^1 f^2 f^3 f^4}}{\lambda f^i} \,. \tag{69}$$

Using the commutativity of the partial derivatives

$$\frac{\partial}{\partial x^j}\frac{\partial\tilde{S}}{\partial x^i} = \frac{\partial}{\partial x^i}\frac{\partial\tilde{S}}{\partial x^j} \tag{70}$$

we get the system of six differential equations for $\lambda(x)$. Writing down one of them for $i = 1$, $j = 2$, one gets:

$$3(f^1)^2\frac{\partial\lambda}{\partial x^1} - 3(f^2)^2\frac{\partial\lambda}{\partial x^2} = \lambda(f^1 - f^2) \,. \tag{71}$$

If $\lambda = const$, then $f^i = f^j = const$, which means that $\tilde{S}$ is a following linear function of coordinates:

$$\tilde{S} = a\left(x^1 + x^2 + x^3 + x^4\right) + b \,, \tag{72}$$

where a, b are constants.

If $\lambda \neq const$ and $f^i \neq 0$, then we introduce the following notation for the indefinite integrals

$$I^i = \int^{x^i} \frac{dx^{i-}}{(f^{i-})^2} \,, \qquad J^i = \int^{x^i} \frac{dx^{i-}}{3f^{i-}} \,, \tag{73}$$

and the equation (71) and its analogues give

$$\lambda(x^1, x^2, x^3, x^4) = \exp\left(W(I^1 + I^2 + I^3 + I^4) + J^1 + J^2 + J^3 + J^4\right) \,, \tag{74}$$

where W is an arbitrary function of a single real variable. The World function, $\tilde{S}$, can be obtained with the help of a line integral of the second kind for an arbitrary path in the H_4 space. This path connects the fixed point with the point $M(x^1, x^2, x^3, x^4)$.

The expressions (69), (73) and (74) mean that the derivatives $\frac{\partial\tilde{S}}{\partial x^i}$ are not the components of the analytical function of the variables H_4 or their linear combinations. The only exception takes place when all these derivarives are equal and equal to a constant a (72). The same can be stated for the function, $\tilde{S}$, if we exclude the linear dependence (72). But for every analytical function, $F(X)$, with $f^i \neq 0$, there is a corresponding World function, $\tilde{S}$, that can be expressed with the help of the squares of the components of $F(X)$, while the corresponding field of velocities defining the world lines is an analytical function, $F(X)$, of variables H_4.

Hypothesis II_{H_4}: Components of the vector field that produces the world lines corresponding to the given World function, are the components of the function of the variable H_4 symmetrically conjugate to the analytical function of the same variable.

According to (52), (54), (58) the symmetrical conjugation (40) in the H_4 space coincide with the normal conjugation [9], and in the mentioned special basis the expression (40) becomes

$$y^i = \frac{x^1 x^2 x^3 x^4}{x^i}\,. \tag{75}$$

Taking into acount this formula and the expression (65) as a consequence of the Hypothesis II, one gets

$$\frac{f^1 f^2 f^3 f^4}{f^i} = \frac{\frac{\partial \tilde{S}}{\partial x^1}\frac{\partial \tilde{S}}{\partial x^2}\frac{\partial \tilde{S}}{\partial x^3}\frac{\partial \tilde{S}}{\partial x^4}}{\frac{\partial \tilde{S}}{\partial x^i}} \cdot \lambda(x)^3\,, \tag{76}$$

or

$$f^i(x^{i-}) = \frac{\partial \tilde{S}}{\partial x^i} \cdot \lambda(x)\,. \tag{77}$$

If $\lambda = const$, then

$$\tilde{S} = \frac{1}{4}\left(\tilde{f}^1(x^{1-}) + \tilde{f}^2(x^{2-}) + \tilde{f}^3(x^{3-}) + \tilde{f}^4(x^{4-})\right)\,, \tag{78}$$

where $\tilde{f}^i(x^{i-})$ is a function of a single real variable x^{i-}. Within the accuracy of a number factor, these functions are the primitives of the components $f^i(x^{i-})$ of the initial analytical functions $F(X)$. The properties of the polynumbers H_4 provide the formal coincidence of the scalar function, $\tilde{S}$, (78), with the component of the analytical function

$$\tilde{F}(X) = \tilde{f}^i(x^{i-})e_i \tag{79}$$

for the unity element in the basis $1, j, k, jk$; $j^2 = k^2 = (jk)^2 = 1$:

$$\left.\begin{array}{ll} 1 = e_1 + e_2 + e_3 + e_4\,, & j = e_1 + e_2 - e_3 - e_4\,, \\ k = e_1 - e_2 + e_3 - e_4\,, & jk = e_1 - e_2 - e_3 + e_4\,. \end{array}\right\} \tag{80}$$

Let $\lambda \neq const$, then the expression (77) gives the system of six equations to define function $\lambda(x)$:

$$f^i \frac{\partial \lambda}{\partial x^j} = f^j \frac{\partial \lambda}{\partial x^i} \tag{81}$$

The general solution of this system is

$$\lambda(x) = \Lambda\left(\tilde{f}^1(x^{1-}) + \tilde{f}^2(x^{2-}) + \tilde{f}^3(x^{3-}) + \tilde{f}^4(x^{4-})\right)\,, \tag{82}$$

where Λ is a function of a single real variable, and $\tilde{f}^i(x^{i-})$ are the primitives of the components $f^i(x^{i-})$ of the initial analytical function $F(X)$.

The World function $\tilde{S}$ can be obtained with the help of a line integral of the second kind for an arbitrary path in the H_4 space. This path connects the fixed point with the point $M(x^1, x^2, x^3, x^4)$.

In general case, the derivatives $\frac{\partial \tilde{S}}{\partial x^i}$ are not the components of the analytical function of the variable H_4 or their linear combinations. The same can be stated for function $\tilde{S}$. But for every analytical function $F(X)$ there is a corresponding World function, $\tilde{S}$, that can be expressed with the help of the squares of the components of $F(X)$, while the corresponding field of velocities defining the world lines is symmetrically conjugate to the analytical function $F(X)$ of variables H_4.

Let us suggest that we know the World function in the space (63) which is conformly connected to the Berwald-Moor space. Let us regard tensor

$$g^{ij}(x) = \frac{1}{\kappa(x)^2 \cdot \mu^2 \cdot (mc)^2} \overset{o}{g}{}^{ijkm} \frac{\partial \tilde{S}}{\partial x^k} \frac{\partial \tilde{S}}{\partial x^m}, \tag{83}$$

in which $\mu = 1/4$ according to (57). Let $det(g^{ij}(x)) \neq 0$, then in the same coordinate space, x^1, x^2, x^3, x^4, one can define the pseudo Riemannian geometry with the length element

$$ds' = \kappa(x) \cdot \mu \cdot mc\sqrt{g_{ij} dx^i dx^j} \tag{84}$$

and the tangent equation for the indicatrix

$$g^{ij} p'_i p'_j = \kappa(x)^2 \cdot \mu^2 \cdot (mc)^2 . \tag{85}$$

The Hamilton-Jacoby equation for the action, S', is

$$g^{ij} \frac{\partial S'}{\partial x^i} \frac{\partial S'}{\partial x^j} = \kappa(x)^2 \cdot \mu^2 \cdot (mc)^2 , \tag{86}$$

and the field of velocities defining the congruence of the world lines has the form

$$\dot{x}^i = g^{ij} \frac{\partial S'}{\partial x^j} \lambda'(x) , \tag{87}$$

where $\lambda'(x)$ is a scalar function. Substituting the expression for g^{ij} (83) into the last two formulas, one can see that the solution of the equation (86) is the World function $S' = \tilde{S}$, and the congruences of the world lines in the spaces (63) and (84) coincide.

Let us regard tensor

$$G^{ij}(x) = \overset{o}{g}{}^{ijkm} \frac{\partial \tilde{S}}{\partial x^k} \frac{\partial \tilde{S}}{\partial x^m} , \tag{88}$$

which coincides with tensor g^{ij} (83) within the accuracy of a number factor. In the matrix form

$$\left(G^{ij}(x)\right) = \frac{1}{12} \begin{pmatrix} 0 & \frac{\partial \tilde{S}}{\partial x^3}\frac{\partial \tilde{S}}{\partial x^4} & \frac{\partial \tilde{S}}{\partial x^2}\frac{\partial \tilde{S}}{\partial x^4} & \frac{\partial \tilde{S}}{\partial x^2}\frac{\partial \tilde{S}}{\partial x^3} \\ \frac{\partial \tilde{S}}{\partial x^3}\frac{\partial \tilde{S}}{\partial x^4} & 0 & \frac{\partial \tilde{S}}{\partial x^1}\frac{\partial \tilde{S}}{\partial x^4} & \frac{\partial \tilde{S}}{\partial x^1}\frac{\partial \tilde{S}}{\partial x^3} \\ \frac{\partial \tilde{S}}{\partial x^2}\frac{\partial \tilde{S}}{\partial x^4} & \frac{\partial \tilde{S}}{\partial x^1}\frac{\partial \tilde{S}}{\partial x^4} & 0 & \frac{\partial \tilde{S}}{\partial x^1}\frac{\partial \tilde{S}}{\partial x^2} \\ \frac{\partial \tilde{S}}{\partial x^2}\frac{\partial \tilde{S}}{\partial x^3} & \frac{\partial \tilde{S}}{\partial x^1}\frac{\partial \tilde{S}}{\partial x^3} & \frac{\partial \tilde{S}}{\partial x^1}\frac{\partial \tilde{S}}{\partial x^2} & 0 \end{pmatrix} . \tag{89}$$

Since

$$det\left(G^{ij}\right) = -\frac{3}{12^4} \left(\frac{\partial \tilde{S}}{\partial x^1} \frac{\partial \tilde{S}}{\partial x^2} \frac{\partial \tilde{S}}{\partial x^3} \frac{\partial \tilde{S}}{\partial x^4} \right)^2 \neq 0 , \tag{90}$$

due to the inequality (61), one can construct tensor G_{ij}, and, thus, construct tensor g_{ij}.

The basis, e_1, e_2, e_3, e_4, used in this Section is not the physical basis commonly used. So, let us pass to the basis (80), though not for the general case but for the simplest World function

$$\tilde{S} = \frac{1}{4}\left(x^1 + x^2 + x^3 + x^4\right) + const , \tag{91}$$

which in the basis (80) has the form

$$\tilde{S} = x^0 + const , \tag{92}$$

where x^0 is the coordinate of the unity element in the basis (80). In this case matrix (G^{ij}) takes the form

$$\left(G^{ij}(x)\right) = \frac{1}{12\cdot 4^2}\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}. \tag{93}$$

To obtain matrix (G^{ij}) of tensor G^{ij} in the new basis (80), that is matrix $\left(G^{i'j'}\right)$, one should multiply the matrix (G^{ij}) (with regard to the fact that the transition matrix is symmetrical) from the left and from the right by the matrix reverse to the transition matrix. The result is

$$\left(G^{i'j'}(x)\right) = \frac{1}{4^4}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{3} & 0 & 0 \\ 0 & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & -\frac{1}{3} \end{pmatrix}. \tag{94}$$

Thus, the World function (91) in space H_4 corresponds to the pseudo Euclidean geometry with the signature $(1,-1,-1,-1)$.

Conclusion

All the above said means that the relation between the World function, $\tilde{S}$, defined in a polynumber space P_n, and the analytical functions of the variable P_n can be postulated in various forms.

The most strong limitations on the form of the World function, $\tilde{S}$, are given by *Hypothesis I*: *Components of the vector field that produces the world lines corresponding to the given World function, are the components of the analytical function of the variable* P_n.

Less strong though strong enough limitations on the form of the World function, $\tilde{S}$, are given by *Hypothesis II*: *Components of the vector field that produces the world lines corresponding to the given World function, are the components of the function of the variable* P_n *symetrically conjugate to the analytical function of the same variable.*

It seems that *Hypothesis II* is more closely linked to Physics.

Although the approach used to describe the World with the help of a World function demands some operation of the "index rising" for the covariant tensors (and this operation can be always realized for a fixed geometry), the all-sufficient pair {World function; congruence of the world lines} can correspond to qualitatively different geometries.

In this paper it is shown that Finsler space H_4 with the Berwald-Moor metric corresponds to the Minkowski space.

Finally, regarding the physical World as the congruence of the world lines in the four dimensional space-time, we conclude that the geometry is not a fixed notion. One can pass from one geometry to another depending on the problems of interest, and with this not only the congruence of the world lines, i.e. World itself, will be conserved, but the World function also.

Thus Minkowskian space and polynumber space H_4 correspond to the same physical World.

References

[1] H. Weyl. Space, Time, Matter. Janus, Moscow, 1996.

[2] P.K. Rashevskii. Riemannian geometry and tensor analysis. Nauka, Moscow, 1967.

[3] M. Matsumoto, S. Numata: On Finsler spaces with 1-form metric. Tensor, N.S., Vol. 32 (1978), 161.

[4] M. Matsumoto, S. Numata: On Finsler spaces with 1-form metric II. Berwald - Moor's metric., Tensor, N.S., Vol. 32 (1978), 275.

[5] M. Matsumoto, S. Numata: On Finsler spaces with a cubic metric, Tensor, N.S., Vol. 33 (1979), 153.

[6] H. Shimada: On Finsler spaces with the metric L of m-th root metric, Tensor, N.S., Vol. 33 (1979), 365 - 372.

[7] D.P. Pavlov. Generalized axioms of scalar product. Hypercomplex Numbers in Geometry and Physics, Vol.1(2204), 5-19.

[8] G.I. Garas'ko. Generalized analytical functions of the polynumber variable. Hypercomplex Numbers in Geometry and Physics, Vol.1 (2204), 75-88.

[9] D.P. Pavlov, G.I. Garas'ko. Normal conjugation on the set of polynumbers. Hypercomplex Numbers in Geometry and Physics, Vol. 2 (2204), 6-14.