INVARIANT FRAMES FOR A GENERALIZED LAGRANGE SPACE WITH BERWALD-MOÓR METRIC

Marius Păun

Dept of Algebra and Geometry, Faculty of Mathematics and Informatics, Transilvania, University of Brasov, Braşov, România

m.paun@unitbv.ro

The notion of generalized Lagrange space should be geometrically considered as a generalized metric space $M^n = (M, g_{ij}(x,y))$. A theory of invariant Finsler spaces was given by M. Matsumoto and R. Miron with important applications. The notion of non-holonomic space was introduced by Gh. Vranceanu in [VR]. The Vranceanu type invariant frames and the invariant geometry of second order Lagrange spaces was studied by the author in [P3]. The purpose of the present paper is to study the invariant geometry for a generalized Lagrange space endowed with a Berwald-Moór metric. We introduce distinct non-holonomic frames on the two components of the Whitney's decomposition. This will determine a non-holonomic coordinates system on the total space TM and thus its geometry can be studied with methods analogous to the mobile frame. We obtain, in this manner, invariant connections, curvatures and torsions, and the fundamental equations in this theory. Also we can construct the invariant frames so that, with respect to them, the metric of the total space can be written in canonical form and in this case we deduce invariant Einstein equations. We mention that the frames introduced here depend on the metric and all the computations are for this metric.

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1 General invariant frames

Let M be a 4-dimensional C^∞ class manifold, (TM, π, M) its tangent bundle, (x^i, y^i) local coordinates on TM, $F : TM \to R$, $F = F(y)$ a locally Minkowsky Finsler function and the induced fundamental metric tensor

$$g^*_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}, \qquad i,j = 1,2,3,4.$$

For $F(y) = \sqrt[4]{y^1 y^2 y^3 y^4}$ Pavlov has studied the "4-pseudo scalar product"

$$< X, Y, Z, T > = G_{ijkl} X^i Y^j Z^k T^l \tag{1.1}$$

related to the Berwald-Moór metric, where

$$G_{ijkl} = \frac{1}{4!}\frac{\partial^4 \mathcal{L}}{\partial y^i \partial y^j \partial y^k \partial y^l}, \quad \text{and} \quad \mathcal{L} = F^4.$$

Based on the tensor field (1.1) Balan and Brinzei have constructed an energy-dependent metric space which is a generalized Lagrange space (which is a Euclidean-locally Minkowsky relativistic model).

The canonical nonlinear connection N on TM, in this case, has the coefficients $N^i_j = 0$ so the adapted basis to the direct decomposition

$$T_u(TM) = N(u) \oplus V(u) \ \ \forall u \in TM, \tag{1.2}$$

is, in fact, the canonical one:

$$\left\{\frac{\delta}{\delta x^i},\ \frac{\delta}{\delta y^i}\right\} = \left\{\frac{\partial}{\partial x^i},\ \frac{\partial}{\partial y^i}\right\} \qquad (i = 1,2,3,4) \tag{1.3}$$

The invariant frames adapted to the direct decomposition (1.2)

$$\mathcal{R} = (e^{(0)i}_{\alpha},\ e^{(1)i}_{\alpha}),$$

where i is a component index and α counter index, are defined as follows:

$$e^{(0)}_{\alpha} : u \in TM \to e^{(0)}_{\alpha}(u) \subset N(u) \tag{1.4}$$

$$e^{(1)}_{\alpha} : u \in TM \to e^{(1)}_{\alpha}(u) \subset V(u) \quad (\alpha = 1,2,3,4)$$

For this frames we have $e^{(A)}_{\alpha}(u) = e^{(A)i}_{\alpha}\frac{\delta}{\delta y^{(A)i}}\ |_u$ where, for simplicity, we put $y^{(0)i} = x^i,\ \ y^{(1)i} = y^i$. Denote by $\mathcal{R}^* = (f^{(0)\alpha}_i,\ f^{(1)\alpha}_i)$ the dual frames of $\mathcal{R}$. For $\mathcal{R}^*$, i is the counter index and α is a component index.

The duality conditions are:

$$\langle\ e^{(A)i}{}_{\alpha},\ f^{(B)\alpha}{}_{j}\ \rangle\ =\ \delta^i_j\,\delta^A_B \quad (\,A,B = 0,1\,) \tag{1.5}$$

The frames $\mathcal{R}$ and $\mathcal{R}^*$ are non-holonomic and thru them we can introduce a non-holonomic coordinate system ($s^{(0)\alpha}$,$s^{(1)\alpha}$) in Vrânceanu sense. In this frames the basis and the cobasis have the representations:

$$\begin{aligned}\frac{\delta}{\delta x^i} &= f^{(0)\alpha}{}_i\frac{\delta}{\delta s^{(0)\alpha}} \quad ;\frac{\delta}{\delta y^i} = f^{(1)\alpha}{}_i\frac{\delta}{\delta s^{(1)\alpha}},\\ \delta x^i &= e^{(0)i}{}_{\alpha}\delta s^{(0)\alpha} \quad ;\delta y^i = e^{(1)i}{}_{\alpha}\delta s^{(1)\alpha};\end{aligned} \tag{1.6}$$

These relations hold:

$$\left\langle\frac{\delta}{\delta s^{(A)\alpha}}\,,\,\delta s^{(B)\beta}\right\rangle\ =\ \delta^{\beta}_{\alpha}\,\delta^B_A, \qquad (A,B\ =\ 0,1\,) \tag{1.7}$$

The Lie brackets introduce us the non-holonomy coefficients of Vranceanu:

$$\left[\frac{\delta}{\delta s^{(A)\alpha}}\ ,\ \frac{\delta}{\delta s^{(B)\beta}}\right]\ =\ \underset{(AB)}{\overset{C}{W}}{}^{\gamma}_{\alpha\beta}\,\frac{\delta}{\delta s^{(C)\gamma}} \tag{1.8}$$

$(A,B,C = 0,1;\quad A \leq B;\quad sum\ on\ C)$ where:

$$\underset{(00)}{\overset{0}{W}}{}^{\gamma}_{\beta\alpha} = f^{(0)\gamma}{}_l\left(\frac{\delta e^{(0)l}{}_{\beta}}{\delta s^{(0)\alpha}} - \frac{\delta e^{(0)l}{}_{\alpha}}{\delta s^{(0)\beta}}\right);\quad \underset{(00)}{\overset{1}{W}}{}^{\gamma}_{\beta\alpha} = 0 \tag{1.9}$$

$$\underset{(01)}{\overset{0}{W}}{}^{\gamma}_{\beta\alpha} = f^{(0)\gamma}{}_l\frac{\delta e^{(0)l}{}_{\alpha}}{\delta s^{(1)\beta}};\qquad \underset{(01)}{\overset{1}{W}}{}^{\gamma}_{\beta\alpha} = -f^{(1)\gamma}{}_l\frac{\delta e^{(1)l}{}_{\beta}}{\delta s^{(0)\alpha}}; \tag{1.10}$$

$$\overset{0}{\underset{(11)}{W}}{}^{\gamma}_{\beta\alpha} = 0; \quad \overset{1}{\underset{(11)}{W}}{}^{\gamma}_{\beta\alpha} = f^{(1)\gamma}_{\ \ l}\left(\frac{\delta e^{(1)l}_{\ \ \beta}}{\delta s^{(1)\alpha}} - \frac{\delta e^{(1)l}_{\ \ \alpha}}{\delta s^{(1)\beta}}\right); \tag{1.11}$$

We observe that $\overset{A}{\underset{(AA)}{W}}{}^{\gamma}_{\beta\alpha}$ are tensors and $\overset{0}{\underset{(01)}{W}}{}^{\gamma}_{\beta\alpha}$, $\overset{1}{\underset{(01)}{W}}{}^{\gamma}_{\beta\alpha}$ are non-holonomic objects.

The objects $\overset{A}{\underset{(BC)}{W}}{}^{\gamma}_{\beta\alpha}$ are called Vranceanu's non-holonomy coefficients.

Theorem 1.1 *If $\mathcal{R}$ are normal frames ($e^{(0)i}_{\ \ \alpha} = e^{(1)i}_{\ \ \alpha} = e^{i}_{\alpha}$) then $\mathcal{R}$ is holonomic if and only if the entries of the matrix e^i_α are the gradients of n functions on M.*

The frame $\mathcal{R}$ is holonomic if and only if there exists a set of n functions φ defined on M for which we have

$$f^{\alpha}_{i} = \frac{\partial \varphi^{\alpha}}{\partial x^{i}}$$

if and only if the 1-form $ds^{(0)\alpha} = f^{\alpha}_{i} dx^{i}$ is exact.

Observation 1.1 *$\mathcal{R}$ are holonomic if and only if the Lie brackets are vertical i.e. N is integrable, condition which is independent of the frames$\mathcal{R}$.*

Generally we have:

Theorem 1.2 *The frames $\mathcal{R}$ are holonomic if and only if :*

$$\overset{0}{\underset{(00)}{W}}{}^{\gamma}_{\beta\alpha} = \overset{0}{\underset{(01)}{W}}{}^{\gamma}_{\beta\alpha} = \overset{0}{\underset{(11)}{W}}{}^{\gamma}_{\beta\alpha} = 0$$

2 The representation of geometric objects in the frames $\mathcal{R}$

Let $X \in \chi(E)$ be a vector field. Then for X we have the local representation:

$$X = X^{(0)i}\frac{\delta}{\delta x^{i}} + X^{(1)i}\frac{\delta}{\delta y^{i}} \tag{2.1}$$

and the representation:

$$X = X^{(0)\alpha}\frac{\delta}{\delta s^{(0)\alpha}} + X^{(1)\alpha}\frac{\delta}{\delta s^{(1)\alpha}} \tag{2.2}$$

in invariant frames. The relations between the components in the local basis and the non-holonomic components of the vector field X are:

$$X^{(A)i} = e^{(A)i}_{\alpha} X^{(A)\alpha} \quad or \quad X^{(A)\alpha} = f^{(A)\alpha}_{i} X^{(A)i} \quad (A = 0, 1). \tag{2.3}$$

Proposition 2.1 *The non-holonomic components of the h_{-}, v_{-} projections of the vector field X are invariant at local changes of coordinates.*

We can easy see that from:

$$\bar{X}^{(A)\alpha} = \bar{f}^{(A)\alpha}_{\ \ i}\bar{X}^{(A)i} = f^{(A)\alpha}_{\ \ k}\frac{\partial \bar{x}^k}{\partial x^i}X^{(A)l}\frac{\partial x^i}{\partial \bar{x}^l} = f^{(A)\alpha}_{\ \ k}\delta^k_l X^{(A)l} = X^{(A)\alpha} \ \blacksquare$$

Let $\omega \in \chi^*$ be a field of 1-forms. Then:

$$\omega = \omega^{(0)}_{\ \ i}\delta x^i + \omega^{(1)}_{\ \ i}\delta y^i = \omega^{(0)}_{\ \ \alpha}\delta s^{(0)\alpha} + \omega^{(1)}_{\ \ \alpha}\delta s^{(1)\alpha} \tag{2.4}$$

Proposition 2.2 *The non-holonomic components $\omega^{(A)}_{\ \ \alpha}$ of the projections of $\omega^{(A)}$ are invariant at local changes of coordinates.*

We can use the same way to define the non-holonomic components of a tensor field and to prove that there are invariant at local changes of coordinates. Due to the fact that the geometric objects studied above have, in the frames $\mathcal{R}$, invariant components at local changes of coordinates we call this frames invariant frames.

3 d-linear connections in invariant frames

Since F is locally Minkowsky the coefficients of the canonical linear d-connection $C\Gamma(N)$ are:

$$L^i_{jk} = 0; \qquad C^i_{jk} = \frac{1}{2}g^{ih}\left(\frac{\partial g_{jh}}{\partial y^k} + \frac{\partial g_{kh}}{\partial y^j} - \frac{\partial g_{jk}}{\partial y^h}\right) \tag{3.1}$$

Proposition 3.1 *In the invariant frames $\mathcal{R}$ the essential components of the canonical d-linear connection $\mathcal{C}\Gamma(N)$ are four and are given by:*

$$\overset{A}{L}{}^{\gamma}_{\beta\alpha} = f^{(A)\gamma}_{\ \ m}\frac{\delta e^{(A)m}_{\ \ \ \beta}}{\delta s^{(0)\alpha}} \quad (A = 0, 1) \tag{3.2}$$

$$\overset{A}{C}{}^{\gamma}_{\beta\alpha} = f^{(A)\gamma}_{\ \ m}\left(\frac{\delta e^{(A)m}_{\ \ \ \beta}}{\delta s^{(1)\alpha}} + e^{(1)i}_{\ \ \alpha}e^{(A)j}_{\ \ \beta}C^m_{ij}\right) \quad (A = 0, 1)$$

i.e.

$$\mathcal{C}\Gamma(N) = \left(\frac{1}{2}\underset{(00)}{\overset{0}{W}}{}^{\gamma}_{\beta\alpha}\ ,\ -\frac{1}{2}\underset{(01)}{\overset{1}{W}}{}^{\gamma}_{\beta\alpha}\ ,\ \overset{0}{C}{}^{\gamma}_{\beta\alpha}\ ,\ \overset{1}{C}{}^{\gamma}_{\beta\alpha}\right) \tag{3.3}$$

In the case of the normal invariant frames the canonical linear d-connection has only two distinct coefficients:

$$\mathcal{C}^*\Gamma(N) = \left(L^{\gamma}_{\beta\alpha},\ C^{\gamma}_{\beta\alpha}\right) = \left(\frac{1}{2}\underset{(00)}{\overset{0}{W}}{}^{\gamma}_{\beta\alpha}\ ,\ \frac{1}{2}\underset{(11)}{\overset{1}{W}}{}^{\gamma}_{\beta\alpha} + f^{\gamma}_m e^i_{\beta}e^j_{\alpha}C^m_{ij}\right) \tag{3.4}$$

Denoting by " $_{|}$"and "|"the covariant derivatives with respect to $C\Gamma(N)$ we have:

Proposition 3.2 *The movement equations of the frames $\mathcal{R}$ and $\mathcal{R}^*$ are:*

$$e^{(A)i}_{\ \ \alpha|m} = \overset{A}{L}{}^{\gamma}_{\beta\alpha}\, e^{(A)i}_{\ \ \gamma} f^{(0)\beta}_{\ \ m} \quad ; \quad e^{(A)i}_{\ \ \alpha}|_m = \overset{A}{C}{}^{\gamma}_{\beta\alpha}\, e^{(A)i}_{\ \ \gamma} f^{(1)\beta}_{\ \ m} \tag{3.5}$$

$$f^{(A)\gamma}_{\ \ i|m} = -\overset{A}{L}{}^{\gamma}_{\beta\alpha}\, f^{(A)\alpha}_{\ \ i} f^{(0)\beta}_{\ \ m} \quad ; \quad f^{(A)\alpha}_{\ \ i}| = -\overset{A}{C}{}^{\gamma}_{\beta\alpha}\, f^{(A)\alpha}_{\ \ i} f^{(1)\beta}_{\ \ m} \qquad (A = 0, 1) \tag{3.6}$$

Proposition 3.3 *If we have the canonical d-linear connection $C\Gamma(N)$ and the invariant frames $\mathcal{R}$ then the coefficients of $\mathcal{C}\Gamma(N)$ can be expressed in the shape:*

$$\overset{A}{L}{}^{\gamma}_{\beta\alpha} = \frac{\delta e^{(A)m}_{\ \ \beta}}{\delta x^i} e^{(0)i}_{\ \ \alpha} f^{(A)\gamma}_{\ \ m}; \tag{3.7}$$

or

$$\overset{A}{L}{}^{\gamma}_{\beta\alpha} = -e^{(0)i}_{\ \ \alpha} \frac{\delta f^{(A)\gamma}_{\ \ i}}{\delta x^j} e^{(A)j}_{\ \ \beta};$$

or:

$$\overset{A}{L}{}^{\gamma}_{\beta\alpha} = e^{(A)k}_{\ \ \beta|i} e^{(0)i}_{\ \ \alpha} f^{(A)\gamma}_{\ \ k} = -f^{(A)\gamma}_{\ \ k|i} e^{(A)k}_{\ \ \beta} e^{(0)i}_{\ \ \alpha}$$

$$\overset{A}{C}{}^{\gamma}_{\beta\alpha} = \left(\frac{\delta e^{(A)m}_{\ \ \beta}}{\delta y^i} + e^{(A)j}_{\ \ \beta}\, C^m_{ij} \right) e^{(1)i}_{\ \ \alpha} f^{(A)\gamma}_{\ \ m}; \tag{3.8}$$

or:

$$\overset{A}{C}{}^{\gamma}_{\beta\alpha} = e^{(0)i}_{\ \ \alpha} \left(-\frac{\delta f^{(A)\gamma}_{\ \ i}}{\delta y^j} + C^m_{ij}\ f^{(A)\gamma}_{\ \ m} \right) e^{(A)j}_{\ \ \beta};$$

or:

$$\overset{A}{C}{}^{\gamma}_{\beta\alpha} = e^{(A)k}_{\ \ \beta}|_i e^{(0)i}_{\ \ \alpha} f^{(A)\gamma}_{\ \ k} = -f^{(A)\gamma}_{\ \ k}|_i e^{(a)k}_{\ \ \beta} e^{(0)i}_{\ \ \alpha}$$

4 Covariant invariant derivatives

Consider the d-linear connection whose coefficients in the frames $\mathcal{R}$ are

$$\mathcal{C}\Gamma(N) = \left(\overset{(A)}{L}{}^{\gamma}_{\ \beta\alpha},\ \overset{(A)}{C}{}^{\gamma}_{\ \beta\alpha} \right) \quad (A = 0, 1)$$

Let $X \in \chi(E)$ and $X^{(0)\alpha}, X^{(1)\alpha}$ its invariant components. We denote by " $)$", and " $)$ ",

the operators of h_- and v_- covariant invariant derivative acting in this manner:

Definition 4.1 *The h_- and v_- invariant covariant derivatives of the components $X^{(0)}, X^{(1)}$ are*

$$X^{(A)\alpha}{}_{)\beta} = \frac{\delta X^{(A)\alpha}}{\delta s^{(0)\beta}} + \overset{A}{L}{}^{\alpha}{}_{\varphi\beta} X^{(A)\varphi} \tag{4.9}$$

$$X^{(A)\alpha}\,)_{\beta} = \frac{\delta X^{(A)\alpha}}{\delta s^{(1)\beta}} + \overset{A}{C}{}^{\alpha}{}_{\varphi\beta} X^{(A)\varphi} \quad ; \quad (A = 0, 1) \tag{4.10}$$

If ω is an 1-form field and $\omega^{(A)}{}_{\alpha}$, $(A = 0, 1)$ are its invariant components then:

Definition 4.2 *The h_-, v_- covariant invariant derivatives of the components $\omega^{(0)}{}_{\alpha}$, $\omega^{(1)}{}_{\alpha}$ are*

$$\omega^{(A)}{}_{\alpha)\beta} = \frac{\delta \omega^{(A)}{}_{\alpha}}{\delta s^{(0)\beta}} - \overset{A}{L}{}^{\varphi}{}_{\alpha\beta}\, \omega^{(A)}{}_{\varphi} \tag{4.11}$$

$$\omega^{(A)}{}_{\alpha}\,)_{\beta} = \frac{\delta \omega^{(A)}{}_{\alpha}}{\delta s^{(1)\beta}} - \overset{A}{C}{}^{\varphi}{}_{\alpha\beta}\, \omega^{(A)}{}_{\varphi} \quad ; \quad (A = 0, 1) \tag{4.12}$$

Theorem 4.1 *The h_-, v_- covariant invariant derivatives of $X^{(A)\alpha}$ and $\omega^{(A)}_{\alpha}$ are the invariant components of the h_-, v_- covariant derivatives of the components in the canonical basis of X and ω i.e.*

$$X^{(A)\alpha}{}_{)\beta} = X^{(A)i}{}_{|m} f^{(A)\alpha}_{i} e^{(0)m}_{\beta} \tag{4.13}$$

$$X^{(A)\alpha}\,)_{\beta} = X^{(A)i} \mid_{m} f^{(A)\alpha}_{i} e^{(1)m}_{\beta} \tag{4.14}$$

$$\omega^{(A)}{}_{\alpha)\beta} = \omega^{(A)}{}_{i|m} e^{(A)i}_{\alpha} e^{(0)m}_{\beta} \tag{4.15}$$

$$\omega^{(A)}{}_{\alpha}\,)_{\beta} = \omega^{(A)}{}_{i} \mid_{m} e^{(A)i}_{\alpha} e^{(1)m}_{\beta} \tag{4.16}$$

or equivalent:

$$X^{(A)i}{}_{|m} = X^{(A)\alpha}{}_{)\beta} e^{(A)i}_{\alpha} f^{(0)\beta}_{m} \tag{4.17}$$

$$X^{(A)i} \mid_{m} = X^{(A)\alpha})_{\beta} e^{(A)i}_{\alpha} f^{(1)\beta}_{m} \tag{4.18}$$

$$\omega^{(A)}{}_{i|m} = \omega^{(A)}{}_{\alpha)\beta} f^{(A)\alpha}_{i} f^{(0)\beta}_{m} \tag{4.19}$$

$$\omega^{(A)}{}_{i} \mid_{m} = \omega^{(A)}{}_{\alpha})_{\beta} f^{(A)\alpha}_{i} f^{(1)\beta}_{m}. \tag{4.20}$$

Now we can formulate the laws of transformation of the d-linear connection at invariant frames changes.

Proposition 4.1 *At invariant frames changes* $\overset{A}{\bar{L}}{}^{\gamma}_{\alpha\beta}$ *and* $\overset{A}{\bar{C}}{}^{\gamma}_{\alpha\beta}$ *transform like:*

$$\overset{A}{\bar{L}}{}^{\gamma}_{\alpha\beta} = \overset{A}{\bar{C}}{}^{\gamma}_{\psi} \, \overset{A}{C}{}^{\psi}_{\beta)\eta} \, \overset{0}{C}{}^{\eta}_{\alpha} \tag{4.21}$$

$$\overset{A}{\bar{C}}{}^{\gamma}_{\alpha\beta} = \overset{A}{\bar{C}}{}^{\gamma}_{\psi} \, \overset{A}{C}{}^{\psi}_{\beta})_{\eta} \, \overset{1}{C}{}^{\eta}_{\alpha} \tag{4.22}$$

where we have:

$$\frac{\delta}{\delta \bar{s}^{(A)\alpha}} = \overset{A}{C}{}^{\beta}_{\alpha} \frac{\delta}{\delta s^{(A)\beta}} \qquad \delta s^{(A)\alpha} = \overset{A}{\bar{C}}{}^{\alpha}_{\beta} \, \delta \bar{s}^{(A)\beta} \qquad \overset{A}{\bar{C}}{}^{\alpha}_{\beta} \, \overset{B}{C}{}^{\beta}_{\gamma} = \delta^{\alpha}_{\gamma} \delta^{AB}$$

and δ^{α}_{γ}, δ^{AB} *are Kronecker symbols.*

5 Torsion and curvature tensor fields in invariant frames

The torsion tensor field of the considered d-linear connection is given by:

$$\mathcal{T}(X,Y) = D_X Y - D_Y X - [X,Y], \quad \forall X, Y \in \chi(E). \tag{5.1}$$

In the frames $\mathcal{R}$, for the canonical d-linear connection, this tensor fields have some horizontal and vertical components which all vanish except:

$$h\mathcal{T}\left(\frac{\delta}{\delta s^{(0)\alpha}}, \frac{\delta}{\delta s^{(1)\beta}}\right) = \underset{(1)}{\bar{K}}{}^{\gamma}{}_{\beta\alpha} \frac{\delta}{\delta s^{(0)\gamma}} = \overset{0}{C}{}^{\gamma}_{\beta\alpha} + \underset{(01)}{\overset{0}{W}}{}^{\gamma}_{\beta\alpha} \tag{5.2}$$

Theorem 5.1 *The d-tensors defined by 5.2 represent the invariant components of the d-tensor of torsion of the d-linear connection C.*

The curvature tensor fields of the d-linear connection C on TM are given by

$$\Re(X,Y) \;=\; [\; D_X \;,\; D_Y \;]\; Z \;-\; D_{[X,Y]}Z, \quad \forall X,Y,Z \in \chi(TM). \tag{5.3}$$

In the frames $\mathcal{R}$, for the canonical d-linear connection, the curvature tensor fields vanish all except:

$$\Re\left(\frac{\delta}{\delta s^{(1)\alpha}},\ \frac{\delta}{\delta s^{(1)\beta}}\right)\frac{\delta}{\delta s^{(0)\gamma}} = S_{\gamma\ \beta\alpha}^{\ \varphi}\frac{\delta}{\delta s^{(0)\varphi}} \tag{5.4}$$

whose local expressions are:

$$S_{\gamma\ \beta\alpha}^{\ \varphi} = \frac{\delta \overset{0}{C^{\varphi}_{\gamma\beta}}}{\delta s^{(1)\alpha}} - \frac{\delta \overset{0}{C^{\varphi}_{\gamma\alpha}}}{\delta s^{(1)\beta}} + \overset{0}{C^{\eta}_{\gamma\beta}}\ \overset{0}{C^{\varphi}_{\eta\alpha}} - \overset{0}{C^{\eta}_{\gamma\alpha}}\ \overset{0}{C^{\varphi}_{\eta\beta}} + \underset{(11)}{\overset{1}{W^{\psi}_{\beta\alpha}}}\ \overset{(0)}{C^{\varphi}_{\gamma\psi}} \tag{5.5}$$

Theorem 5.2 *The formula 5.5 represents the invariant components at local changes of coordinates of the curvature tensor fields of the d-linear connection C.*

6 Structure equations in invariant frames

We introduce the invariant covariant differential of the vector field X in the shape:

$$DX = \left\{ dX^{(A)\alpha} + X^{(A)\gamma}\ \overset{(A)}{\omega^{\alpha}_{\ \gamma}} \right\}\frac{\delta}{\delta s^{(A)\alpha}} \qquad (A=0,1;\ \text{sum on } A) \tag{6.1}$$

In order to obtain the structure equations of the given d-linear connection in invariant frames, it is necessary to compute the exterior differentials of the 1-forms of connection and of the 1-forms $\delta s^{(A)\alpha}$.

Theorem 6.1 *The exterior differentials of the 1-forms $\delta s^{(A)\alpha}$ (A=0,1) depend only on the non-holonomy coefficients of Vranceanu and are given by:*

$$d(\delta s^{(0)\gamma}) = \frac{1}{2}\ \underset{(00)}{\overset{0}{W^{\gamma}_{\beta\alpha}}}\ \delta s^{(0)\alpha}\wedge\delta s^{(0)\beta} + \underset{(01)}{\overset{0}{W^{\gamma}_{\beta\alpha}}}\ \delta s^{(0)\alpha}\wedge\delta s^{(1)\beta} \tag{6.2}$$

$$d(\delta s^{(1)\gamma}) = \underset{(01)}{\overset{1}{W^{\gamma}_{\beta\alpha}}}\ \delta s^{(0)\alpha}\wedge\delta s^{(1)\beta} + \frac{1}{2}\ \underset{(11)}{\overset{1}{W^{\gamma}_{\beta\alpha}}}\ \delta s^{(1)\alpha}\wedge\delta s^{(1)\beta} \tag{6.3}$$

Using the invariant 1-form of connection

$$\overset{A}{\omega}{}^{\alpha}_{\beta} = \overset{A}{L}{}^{\alpha}_{\beta\gamma}\, \delta s^{(0)\gamma} + \overset{A}{C}{}^{\alpha}_{\beta\gamma}\, \delta s^{(1)\gamma}$$

we prove:

Theorem 6.2 *In the invariant frames $\mathcal{R}$ the structure equations of the canonical d-linear connection are given by the relations:*

$$d(\delta s^{(A)\alpha}) - \delta s^{(A)\beta} \wedge \overset{(A)}{\omega}{}^{\alpha}{}_{\beta} = - \overset{(A)}{\Omega}{}^{\alpha} \tag{6.4}$$

$$d(\overset{(A)}{\omega}{}^{\alpha}{}_{\beta}) - \overset{(A)}{\omega}{}^{\gamma}{}_{\beta} \wedge \overset{(A)}{\omega}{}^{\alpha}{}_{\gamma} = - \overset{(A)}{\Omega}{}^{\alpha}{}_{\beta} \qquad (A = 0, 1), \tag{6.5}$$

where the 2-forms of torsion $\overset{(A)}{\Omega}{}^{\alpha}$ are given by:

$$\overset{(0)}{\Omega}{}^{\alpha} = \underset{(1)}{K}{}^{\alpha}_{\beta\gamma}\, \delta s^{(0)\beta} \wedge \delta s^{(1)\gamma}; \qquad \overset{(1)}{\Omega}{}^{\alpha} = 0 \tag{6.6}$$

and the 2-forms of curvature $\overset{(A)}{\Omega}{}^{\alpha}{}_{\beta}$, are given by:

$$\overset{(A)}{\Omega}{}^{\alpha}{}_{\beta} = \frac{1}{2} S_{\beta}{}^{\alpha}{}_{\varphi\psi}\, \delta s^{(1)\varphi} \wedge \delta s^{(1)\psi} \tag{6.7}$$

7 Einstein equations in invariant frames

The locally Minkowsky model of the $GL^{(4)}$ space permits us to consider the normal invariant frames $\bar{\mathcal{R}}$ i.e.

$$e^{(0)i}_{\alpha} = e^{(1)i}_{\alpha} = e^{i}_{\alpha}$$

so that the quadratic form

$$\omega = g_{ij} \delta y^i \delta y^j$$

has the canonical form

$$\omega = (\omega^1)^2 - (\omega^2)^2 - (\omega^3)^2 - (\omega^4)^2 \tag{7.1}$$

We introduce the tensors of Vranceanu

$$\epsilon_{\alpha\beta} = \begin{cases} 1 & \alpha = \beta = 1 \\ -1 & \alpha = \beta = 2, 3, 4 \\ 0 & \mathit{in\ rest} \end{cases} \tag{7.2}$$

and let $\epsilon^{\gamma\beta}$ so that

$$\epsilon_{\alpha\beta}\epsilon^{\gamma\beta} = \delta^{\gamma}_{\alpha} \tag{7.3}$$

where δ^{γ}_{α} is the Kronecker symbol.

Theorem 7.1 *The frames $\bar{\mathcal{R}}$ are pseudo orthogonal and the Vrânceanu tensors represent the invariant components of g_{ij}.*

Theorem 7.2 *Invariant frames transformations which preserve the canonical form of the metric G together with the composition of transformations is a group isomorphic with the multiplicative group of the pseudo orthogonal matrix*

$$\begin{pmatrix} C^{\alpha}_{\beta} & 0 \\ 0 & C^{\alpha}_{\beta} \end{pmatrix}.$$

If we consider the canonical invariant metrical connection

$$\bar{\mathcal{C}}\Gamma(N) = \left\{L^{\gamma}_{\alpha\beta},\ C^{\gamma}_{\alpha\beta}\ \right\}$$

in the frames $\bar{\mathcal{R}}$ it is obvious that $\epsilon_{\alpha\beta}$ satisfy the Matsumoto axioms and the following relations hold good:

$$\epsilon_{\varphi\beta}L^{\varphi}_{\alpha\gamma} + \epsilon_{\alpha\varphi}L^{\varphi}_{\beta\gamma} = 0 \tag{7.4}$$

$$\epsilon_{\varphi\beta}C^{\varphi}_{\alpha\gamma} + \epsilon_{\alpha\varphi}C^{\varphi}_{\beta\gamma} = 0$$

The computation of the invariant components of Ricci tensor and scalar curvature lead us to:

Theorem 7.3 *In the frames $\bar{\mathcal{R}}$, Einstein equations are:*

$$\begin{array}{ccc} 0 = \kappa T^{H}_{\beta\gamma} & 0 = \kappa T^{M_1}_{\beta\gamma} & 0 = \kappa T^{M_2}_{\beta\gamma} \\ S_{\beta\gamma} - \frac{1}{2}\epsilon_{\beta\gamma}S & = & \kappa T^{V}_{\beta\gamma} \end{array} \tag{7.5}$$

where in the right hand of the equations we have the invariant components of the energy-momentum tensor.

Theorem 7.4 *The conservation laws with respect to $\bar{\mathcal{C}}\Gamma(N)$ are:*

$$\left(S^{\alpha}{}_{\beta} - \frac{1}{2}S\delta^{\alpha}_{\beta}\right))_{\alpha} = 0 \tag{7.6}$$

On TM we consider the invariant normal frames $\mathcal{R} = \{X^i_{\alpha},\ X^i_{\alpha}\ \}$, the duals $\mathcal{R}^* = \{Y^{\alpha}_i,\ Y^{\alpha}_i\}$, $y = y^i\dfrac{\delta}{\delta y^i}$ and the restriction of the energy $\mathcal{E}$ on $TM \setminus \{y = 0\}$. Let

$$l^i = \frac{1}{2}g^{ij}\frac{\partial \mathcal{E}}{\partial y^i}.$$

Then $l_i = g_{ij}l^j$, $l^{\alpha} = l^iY^{\alpha}_i$, $l_{\alpha} = X^i_{\alpha}l_i$.

Theorem 7.5 *In this conditions*

$$l^i = y^i$$

and if

$$e^i_{\alpha} = \frac{1}{\mathcal{E}}\left(X^i_{\alpha} - \left(1 - \frac{1}{\mathcal{E}}\right)l^il_{\alpha}\right)$$

then $\mathcal{R} = (e^i_{\alpha}, e^i_{\alpha})$ are a pseudo orthonormated invariant frames with respect to g_{ij}.

Corollary 7.1 *The invariant components of g_{ij} w.r.t $\mathcal{R}$ are exactly the Vranceanu tensors.*

Corollary 7.2 *In the frames $\mathcal{R}$ every d-linear connection is metrical one.*

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