THE BERWALD-MOOR METRIC IN THE TANGENT BUNDLE OF THE SECOND ORDER

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As an application of the results of the first author obtained in the papers [1] and [2], the geometry of the second order tangent bundle T^2M (or second order jet bundle J_0^2M) endowed with two special types of metrics compatible with the 2-contact structures is studied. The particularity of these two models is that the horizontal and the $v^{(1)}$- part of the metric are both given by the same Riemannian metric (respectively, its horizontal part is Riemannian), while its $v^{(2)}$-part is given by the flag-Finsler Berwald-Moor metric (respectively, the $v^{(1)}$ and $v^{(2)}$- parts are given by the flag-Finsler Berwald-Moor metric, [5]).

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1 The 2-Tangent Bundle T^2M

Let M be a real 4-dimensional manifold of class $\mathcal{C}^\infty$, (T^2M, π^2, M) its second order tangent bundle, [1], and let $\widetilde{T^2M}$ be the space T^2M without its null section. For a point $u \in T^2M$, let $(x^i, y^{(1)i}, y^{(2)i})$ be its coordinates in a local chart.

Let N be a nonlinear connection, [3], [8]- [13], and denote its coefficients by $\left(\underset{1}{N}{}^i_j, \underset{2}{N}{}^i_j\right)$, i, $j = 1, ..., 4$. Then, N determines the direct decomposition

$$T_uT^2M = N_0(u) \oplus N_1(u) \oplus V_2(u),\ \forall u \in T^2M. \tag{1}$$

The adapted basis to (1) is $(\delta_i, \delta_{1i}, \delta_{2i})$ and its dual basis is $(dx^i, \delta y^{(1)i}, \delta y^{(2)i})$, where

$$\begin{cases} \delta_i = \dfrac{\delta}{\delta x^i} = \dfrac{\partial}{\partial x^i} - \underset{1}{N}{}^k_i \dfrac{\partial}{\partial y^{(1)k}} - \underset{2}{N}{}^k_i \dfrac{\partial}{\partial y^{(2)k}} \\ \delta_{1i} = \dfrac{\delta}{\delta y^{(1)i}} = \dfrac{\partial}{\partial y^{(1)i}} - \underset{1}{N}{}^k_i \dfrac{\partial}{\partial y^{(2)k}} \\ \delta_{2i} = \dfrac{\partial}{\partial y^{(2)i}} = \dot{\partial}_{2i}, \end{cases} \tag{2}$$

respectively,

$$\begin{cases} \delta y^{(1)i} = dy^{(1)i} + \underset{1}{M}{}^i_k dx^k \\ \delta y^{(2)i} = dy^{(2)i} + \underset{1}{M}{}^i_k dy^{(1)k} + \underset{2}{M}{}^i_k dx^k, \end{cases} \tag{3}$$

where $\underset{1}{M}{}^i_k, \underset{2}{M}{}^i_k$ are the dual coefficients of the nonlinear connection N.

Then, a vector field $X \in \mathcal{X}(T^2M)$ is represented in the local adapted basis as

$$X = X^{(0)i}\delta_i + X^{(1)i}\delta_{1i} + X^{(2)i}\delta_{2i}, \tag{4}$$

with the three right terms (called *d-vector fields*) belonging to the distributions N, N_1 and V_2 respectively.

A 1-form $\omega \in \mathcal{X}^*(T^2M)$ will be decomposed as

$$\omega = \omega_i^{(0)} dx^i + \omega_i^{(1)} \delta y^{(1)i} + \omega_i^{(2)} \delta y^{(2)i}.$$

Similarly, a tensor field $T \in \mathcal{T}_s^r(T^2M)$ can be split with respect to (1) into components, which will be called *d-tensor fields*.

2 N-linear connections. d-tensors of curvature

An *N-linear connection D*, [1], [2], is a linear connection on T^2M, which preserves by parallelism the distributions N, N_1 and V_2.

An N-linear connection is locally given by its coefficients

$$D\Gamma(N) = \left(\underset{(00)}{L}{}^i{}_{jk}, \underset{(10)}{L}{}^i{}_{jk}, \underset{(20)}{L}{}^i{}_{jk}, \underset{(01)}{C}{}^i{}_{jk}, \underset{(11)}{C}{}^i{}_{jk}, \underset{(21)}{C}{}^i{}_{jk}, \underset{(02)}{C}{}^i{}_{jk}, \underset{(12)}{C}{}^i{}_{jk}, \underset{(22)}{C}{}^i{}_{jk} \right), \tag{5}$$

where

$$\begin{cases} D_{\delta_k}\delta_j = \underset{(00)}{L}{}^i{}_{jk}\delta_i, \; D_{\delta_k}\delta_{1j} = \underset{(10)}{L}{}^i{}_{jk}\delta_{1i}, \; D_{\delta_k}\delta_{2j} = \underset{(20)}{L}{}^i{}_{jk}\delta_{2i} \\ D_{\delta_{1k}}\delta_j = \underset{(01)}{C}{}^i{}_{jk}\delta_i, \; D_{\delta_{1k}}\delta_{1j} = \underset{(11)}{C}{}^i{}_{jk}\delta_{1i}, \; D_{\delta_{1k}}\delta_{2j} = \underset{(21)}{C}{}^i{}_{jk}\delta_{2i} \\ D_{\delta_{2k}}\delta_j = \underset{(02)}{C}{}^i{}_{jk}\delta_i, \; D_{\delta_{2k}}\delta_{1j} = \underset{(12)}{C}{}^i{}_{jk}\delta_{1i}, \; D_{\delta_{2k}}\delta_{2j} = \underset{(22)}{C}{}^i{}_{jk}\delta_{2i} \end{cases}. \tag{6}$$

The curvature of the N-linear connection D,

$$R(X,Y)Z = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]}Z,$$

is completely determined by its components (which are d-tensors) $R(\delta_{\gamma l}, \delta_{\beta k})\delta_{\alpha j}$. Namely, the 2-forms of curvature of an N- linear connection are, [1], [2],

$$\begin{aligned} \underset{(\alpha)}{\Omega}{}^i{}_j = & \frac{1}{2}\underset{(0\alpha)}{R}{}_j{}^i{}_{kl} dx^k \wedge dx^l + \underset{(1\alpha)}{P}{}_j{}^i{}_{kl} dx^k \wedge \delta y^{(1)l} + \\ & + \underset{(2\alpha)}{P}{}_j{}^i{}_{kl} dx^k \wedge \delta y^{(2)l} + \frac{1}{2}\underset{(1\alpha)}{S}{}_j{}^i{}_{kl} \delta y^{(1)k} \wedge \delta y^{(1)l} + \\ & + \underset{(2\alpha)}{Q}{}_j{}^i{}_{kl} \delta y^{(1)k} \wedge \delta y^{(2)l} + \frac{1}{2}\underset{(2\alpha)}{S}{}_j{}^i{}_{kl} \delta y^{(2)k} \wedge \delta y^{(2)l}, \end{aligned} \tag{7}$$

$\alpha = 0,1,2$, where the coefficients $\underset{(0\alpha)}{R}{}_j{}^i{}_{kl}$, $\underset{(\beta\alpha)}{P}{}_j{}^i{}_{kl}$, $\underset{(2\alpha)}{Q}{}_j{}^i{}_{kl}$, $\underset{(\beta\alpha)}{S}{}_j{}^i{}_{kl}$ are d-tensors, named the *d-tensors of curvature* of the N-linear connection D.

3 Metric structures on T^2M

A *Riemannian metric* on T^2M is a tensor field G of type $(0,2)$, which is nondegenerate in each $u \in T^2M$ and positively defined on T^2M.

In this paper, we shall consider only metrics in the form

$$G = \underset{(0)}{g}{}_{ij} dx^i \otimes dx^j + \underset{(1)}{g}{}_{ij} \delta y^{(1)i} \otimes \delta y^{(1)j} + \underset{(2)}{g}{}_{ij} \delta y^{(2)i} \otimes \delta y^{(2)j}, \tag{8}$$

where $\underset{(\alpha)}{g}_{ij} = \underset{(\alpha)}{g}_{ij}(x, y^{(1)}, y^{(2)})$; this is, such that the distributions N, N_1 and V_2 generated by the nonlinear connection N be orthogonal in pairs with respect to G.

Let also

$$F = \sqrt[4]{y^{(1)1}y^{(1)2}y^{(1)3}y^{(1)4}}$$

be the Berwald-Moor Finsler function, [14]- [16], and the generalized Lagrange metrics on M, given by

$$h_{ij} = \frac{1}{12F^4}\frac{\partial^2 F^4}{\partial y^i \partial y^j}, \quad \widetilde{h}_{ij} = \frac{1}{12F^6}\frac{\partial^2 F^4}{\partial y^i \partial y^j}. \tag{9}$$

(h defined above is the same as the one in [5], with the only difference that here we have divided by F^4 or F^6 instead of F^2, in order that the obtained tensors be homogeneous of degree -2, respectively, -4).

In the following, we shall use two particular kinds of metrics on $\widetilde{T^2M}$, namely:

1. $\underset{(0)}{g}_{ij} = \underset{(1)}{g}_{ij} = g_{ij}(x), \quad \underset{(2)}{g}_{ij} = \widetilde{h}_{ij}(y^{(1)}),$

2. $\underset{(0)}{g}_{ij} = g_{ij}(x), \quad \underset{(1)}{g}_{ij} = \underset{(2)}{g}_{ij} = h_{ij}(y^{(1)}),$

$g_{ij}(x)$ being a Riemannian metric on M, and h_{ij}, $\widetilde{h}_{ij}$ as above.

These two examples have an important property, namely, they are compatible to the almost contact structures $\mathbb{F}$ introduced in [1].

An N-linear connection D is called *metrical* if $D_X G = 0$, $\forall X \in \mathcal{X}(T^2M)$. The local expression of this equality is given in [1].

4 The Ricci tensor $Ric\,(D)$

If we consider the Ricci tensor $Ric\,(D)$, as the trace of the linear operator

$$V \mapsto R\,(V, X)\,Y,\ \forall V = V^{(0)i}\delta_i + V^{(1)i}\delta_{1i} + V^{(2)i}\delta_{2i} \in \mathcal{X}\left(T^2M\right), \tag{10}$$

then, [3], the Ricci tensor $Ric\,(D)$ has the following components:

$$\begin{aligned}
Ric\,(D)\left(\frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^i}\right) &= \underset{(00)}{R}{}_{i\ jl}^{\ l} =: R_{ij};\\
Ric\,(D)\left(\frac{\delta}{\delta y^{(1)j}}, \frac{\delta}{\delta x^i}\right) &= -\underset{(10)}{P}{}_{i\ lj}^{\ l} =: -\underset{(10)}{\overset{2}{P}}{}_{ij};\\
Ric\,(D)\left(\frac{\delta}{\delta y^{(2)j}}, \frac{\delta}{\delta x^i}\right) &= -\underset{(20)}{P}{}_{i\ lj}^{\ l} =: -\underset{(20)}{\overset{2}{P}}{}_{ij};\\
Ric\,(D)\left(\frac{\delta}{\delta x^j}, \frac{\delta}{\delta y^{(1)i}}\right) &= \underset{(11)}{P}{}_{i\ jl}^{\ l} =: \underset{(11)}{\overset{1}{P}}{}_{ij};\\
Ric\,(D)\left(\frac{\delta}{\delta y^{(1)j}}, \frac{\delta}{\delta y^{(1)i}}\right) &= \underset{(11)}{S}{}_{i\ jl}^{\ l} =: \underset{(1)}{S}{}_{ij};\\
Ric\,(D)\left(\frac{\delta}{\delta y^{(2)j}}, \frac{\delta}{\delta y^{(1)i}}\right) &= -\underset{(21)}{Q}{}_{i\ lj}^{\ l} =: -\underset{(21)}{\overset{2}{Q}}{}_{ij};\\
Ric\,(D)\left(\frac{\delta}{\delta x^j}, \frac{\delta}{\delta y^{(2)i}}\right) &= \underset{(22)}{P}{}_{i\ jl}^{\ l} =: \underset{(22)}{\overset{1}{P}}{}_{ij};
\end{aligned}$$

$$Ric\,(D)\left(\frac{\delta}{\delta y^{(1)j}},\frac{\delta}{\delta y^{(2)i}}\right) = \underset{(22)}{Q}{}_{i\ jl}^{\ l} =: \overset{1}{\underset{(22)}{Q}}{}_{ij};$$
$$Ric\,(D)\left(\frac{\delta}{\delta y^{(2)j}},\frac{\delta}{\delta y^{(2)i}}\right) = \underset{(22)}{S}{}_{i\ jl}^{\ l} =: \underset{(2)}{S}{}_{ij}.$$

5 Canonical structures

Let $(M,\ g)$ be a Riemannian manifold and T^2M, its second order tangent bundle. **The canonical nonlinear connection** N is defined (cf. with R. Miron and Gh. Atanasiu, [13]) by its dual coefficients

$$\underset{(1)}{M}{}^i_j = \gamma^i_{jk}y^{(1)k}, \quad \underset{(2)}{M}{}^i_j = \frac{1}{2}\left\{\mathbb{C}\left(\gamma^i_{jk}y^{(1)k}\right) + \underset{(1)}{M}{}^i_k\underset{(1)}{M}{}^k_j\right\}, \tag{11}$$

$\gamma^i_{jk} = \gamma^i_{jk}(x)$ being the Christoffel symbols of g and $\mathbb{C} = y^{(1)i}\dfrac{\partial}{\partial x^i} + 2y^{(2)i}\dfrac{\partial}{\partial y^{(1)i}}$.

Let

$$\underset{(1)}{N}{}^i_j = \underset{(1)}{M}{}^i_j,\ \underset{(2)}{N}{}^i_j = \underset{(2)}{M}{}^i_j + \underset{(1)}{M}{}^i_k\underset{(1)}{M}{}^k_j$$

be its (direct) coefficients. Then, the coefficients of the Lie brackets, [1],

$$\begin{gathered}[\delta_{0j},\delta_{0k}] = \underset{(01)}{R}{}^{\ i}_{jk}\delta_{1i} + \underset{(02)}{R}{}^{\ i}_{jk}\delta_{2i}, \quad [\delta_{0j},\delta_{1k}] = \underset{(11)}{B}{}^{\ i}_{jk}\delta_{1i} + \underset{(12)}{B}{}^{\ i}_{jk}\delta_{2i}\\ [\delta_{0j},\delta_{2k}] = \underset{(21)}{B}{}^{\ i}_{jk}\delta_{1i} + \underset{(22)}{B}{}^{\ i}_{jk}\delta_{2i}, \quad [\delta_{1j},\delta_{1k}] = \underset{(12)}{R}{}^{\ i}_{jk}\delta_{2i}\\ [\delta_{1j},\delta_{2k}] = \underset{(21)}{B}{}^{\ i}_{jk}\delta_{2i}, \quad [\delta_{2j},\delta_{2k}] = 0\end{gathered} \tag{12}$$

have the property that

$$\underset{(11)}{B}{}^{\ i}_{jk} = \underset{(22)}{B}{}^{\ i}_{jk} = \gamma^i_{jk}, \underset{(21)}{B}{}^{\ i}_{jk} = \underset{(12)}{R}{}^{\ i}_{jk} = \underset{(22)}{R}{}^{\ i}_{jk} = 0. \tag{13}$$

In this paper, we shall use the metrical N-linear connection introduced by the first author, [1], given by the coefficients:

$$\begin{aligned}
\underset{(00)}{L}{}^{\ i}_{jk} &= \frac{1}{2}\underset{(0)}{g}{}^{il}(\delta_k \underset{(0)}{g}{}_{jl} + \delta_j \underset{(0)}{g}{}_{lk} - \delta_l \underset{(0)}{g}{}_{jk})\\
\underset{(\beta 0)}{L}{}^{\ i}_{jk} &= \underset{(\beta\beta)}{B}{}^{\ i}_{kj} + \frac{1}{2}\underset{(\beta)}{g}{}^{il}(\delta_k \underset{(\beta)}{g}{}_{jl} - \underset{(\beta\beta)}{B}{}^{\ m}_{kj}\underset{(\beta)}{g}{}_{ml} - \underset{(\beta\beta)}{B}{}^{\ m}_{kl}\underset{(\beta)}{g}{}_{jm})\\
\underset{(\delta 1)}{C}{}^{\ i}_{jk} &= \frac{1}{2}\underset{(\delta)}{g}{}^{il}\delta_{1k}\underset{(\delta)}{g}{}_{jl},\ \ (\delta = 0, 2),\\
\underset{(\varepsilon 2)}{C}{}^{\ i}_{jk} &= \frac{1}{2}\underset{(\varepsilon)}{g}{}^{il}\dot{\partial}_{2k}\underset{(\varepsilon)}{g}{}_{jl},\ \ (\varepsilon = 0, 1),\\
\underset{(\beta\beta)}{C}{}^{\ i}_{jk} &= \frac{1}{2}\underset{(\beta)}{g}{}^{il}(\delta_{\beta k}\underset{(\beta)}{g}{}_{jl} + \delta_{\beta j}\underset{(\beta)}{g}{}_{lk} - \delta_{\beta l}\underset{(\beta)}{g}{}_{jk}),\ \delta_{2i} = \partial_{2i},
\end{aligned} \tag{14}$$

where $\beta = 1, 2$.

Then, we have to remark that, taking into account the relations (13), two of the coefficients of the torsion tensor vanish, namely

$$\underset{(21)}{P}{}^{\ i}_{jk} = \underset{(12)}{S}{}^{\ i}_{jk} = 0, \tag{15}$$

where $\underset{(21)}{P}{}^{\ i}_{jk}\delta_{1i} = v_1 T(\delta_{2k},\delta_j)$, $\underset{(12)}{S}{}^{\ i}_{jk}\dot{\partial}_{2i} = v_2 T(\delta_{1k},\delta_{1j})$.

6 The case of the $g-h-h$-metric

Let the metric structure of $\widetilde{T^2M}$ be given by

$$G = g_{ij}(x)dx^i \otimes dx^j + h_{ij}(y^{(1)})\,\delta y^{(1)i} \otimes \delta y^{(1)j} + h_{ij}(y^{(1)})\delta y^{(2)i} \otimes \delta y^{(2)j},$$

where g is a Riemannian metric on M and h is as in (9). Then, G is h-Riemannian and v_1-, v_2-locally Minkovski. In this case, the detailed expressions of the coefficients $D\Gamma(N)$ of the canonical N-linear connection and of its curvatures and torsions are given in [1].

By applying the results in the cited paper and the relation (15), we obtain by a direct computation the following result:

Proposition 1. *The only nonvanishing components of the Ricci tensor $Ric(D)$ of the canonical -linear conenction are*

$$\begin{aligned} Ric\,(D)\left(\frac{\delta}{\delta x^j},\frac{\delta}{\delta x^i}\right) &= \underset{(00)}{R}{}_{i\ jk}^{\ k} =: r_{ij};\\ Ric\,(D)\left(\frac{\delta}{\delta x^j},\frac{\delta}{\delta y^{(1)i}}\right) &= \underset{(11)}{P}{}_{i\ jk}^{\ k} =: \underset{(11)}{\overset{1}{P}}{}_{ij};\\ Ric\,(D)\left(\frac{\delta}{\delta y^{(1)j}},\frac{\delta}{\delta y^{(1)i}}\right) &= \underset{(11)}{S}{}_{i\ jk}^{\ k} =: \underset{(1)}{S}{}_{ij}, \end{aligned}$$

where $r_{ij} = r_{i\ jk}^{k}$ denotes the Ricci tensor of the Levi-Civita connection attached to g.

By applying the results in [3], we can state:

Proposition 2. *The Einstein equations associated to the metrical N- linear connection D are*

$$\begin{aligned} &R_{ij} - \tfrac{1}{2}(r + \underset{(1)}{S})g_{ij} = \kappa \underset{(00)}{\mathcal{T}}{}_{ij};\\ &\underset{(11)}{\overset{1}{P}}{}_{ij} = \kappa \underset{(10)}{\mathcal{T}}{}_{ij};\\ &\underset{(1)}{S}{}_{ij} - \tfrac{1}{2}(r + \underset{(1)}{S})h_{ij} = \kappa \underset{(11)}{\mathcal{T}}{}_{ij},\ \alpha = 1,2;\\ &\underset{(20)}{\mathcal{T}}{}_{ij} = \underset{(01)}{\mathcal{T}}{}_{ij} = \underset{(21)}{\mathcal{T}}{}_{ij} = \underset{(02)}{\mathcal{T}}{}_{ij} = \underset{(12)}{\mathcal{T}}{}_{ij} = \underset{(22)}{\mathcal{T}}{}_{ij} = 0. \end{aligned}$$

7 The case of the $g-g-\widetilde{h}$-metric

Proposition 3. *Now, let the metric structure of $\widetilde{T^2M}$ be given by*

$$G = g_{ij}(x)dx^i \otimes dx^j + g_{ij}(x)\,\delta y^{(1)i} \otimes \delta y^{(1)j} + \widetilde{h}_{ij}(y^{(1)})\delta y^{(2)i} \otimes \delta y^{(2)j},$$

where g is a Riemannian metric on M and $\widetilde{h}$ is as in (9). Then, G is h-, v_1- Riemannian and v_2-locally Minkovski.

In order to determine the components of the Ricci tensor, we first have to compute the coefficients of the canonical N-linear connection in our case. We have:

$$\begin{aligned} &\underset{(00)}{L}{}^{i}_{\ jk} = \gamma^{i}_{\ jk},\ \underset{(10)}{L}{}^{i}_{\ jk} = \underset{(10)}{L}{}^{i}_{\ jk}(x),\ \underset{(20)}{L}{}^{i}_{\ jk} = \underset{(20)}{L}{}^{i}_{\ jk}(x, y^{(1)})\\ &\underset{(01)}{C}{}^{i}_{\ jk} = \underset{(11)}{C}{}^{i}_{\ jk} = 0,\ \underset{(21)}{C}{}^{i}_{\ jk} = \frac{1}{2}\tilde{h}^{il}\delta_{1k}\tilde{h}_{jl}\\ &\underset{(02)}{C}{}^{i}_{\ jk} = \underset{(12)}{C}{}^{i}_{\ jk} = \underset{(22)}{C}{}^{i}_{\ jk} = 0. \end{aligned}$$

Using the expressions above, we obtain

Proposition 4. *All the components of the Ricci tensor of the N-linear connection D vanish, except*

$$Ric\,(D)\left(\frac{\delta}{\delta x^j},\frac{\delta}{\delta x^i}\right) = \underset{(00)}{R}{}_{i\ jl}^{\ l} =: r_{ij},$$

where r_{ij} denotes the Ricci tensor of the Levi-Civita connection of the metric g on M.

As a consequence, the Einstein equations can be written in this case as:

$$r_{ij} - \frac{1}{2} r g_{ij} = \kappa \underset{(00)}{\mathcal{T}}{}_{ij},$$

the other components of the energy-momentum tensor being identically 0. The equations above are exactly the Einstein equations of the Levi-Civita connection ∇ of $g = g(x)$. Obviously, the energy conservation law is satisfied.

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