THE PROLONGATIONS OF A FINSLER METRIC TO THE TANGENT BUNDLE $T^k(M)$ $(k > 1)$ OF THE HIGHER ORDER ACCELERATIONS

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An old problem in differential geometry is that of prolongation of a Riemannian structure $g\left(x\right)$ on a real $n-$dimensional C^{∞}-manifold $M, x \in M$, to the bundle of $k-$jets $\left(J_0^k M, \pi^k, M\right)$ or, equivalently the tangent bundle $\left(T^k M, \pi^k, M\right)$ of the higher order accelerations. The problem belongs to so-called geometry of higher order. It was solved in [18] for $k = 1$ and partially in [19] for $k = 2$. The same problem of prolongation can be considered for a Finslerian structure $F\left(x, y^{(1)}\right)$. In the paper [15] are given these solutions in the general cases, using the Sasaki-Matsumoto $N-$lift (for $k = 2$, see [3] and [6]).

But, the terms of Sasaki-Matsumoto prolongation of a Riemannian metric (or Finslerian metric) to $T^k M$ have not the same physical dimensions because these prolongations is not homogeneous on the fibres of the tangent bundle of order k. This is a disavantage in the study of the geometry of $T^k M$ using the Riemannian metrics determined by these prolongations.

In this paper, only for a Finsler space $F^n = \left(M, F\left(x, y^{(1)}\right)\right)$, we correct this disavantage introducing a new kind of prolongation $\overset{\circ}{\mathbf{G}}$ of the Finsler metric $g_{ab}\left(x, y^{(1)}\right) = \partial^2 F / \partial y^{(1)a} \partial y^{(1)b}$ given by (2.1), which is $0-$homogeneous. Some properties of the Riemannian space $\left(\widetilde{T^k M}, \overset{\circ}{G}\right)$ are studied. The almost $(k-1)\, n-$contact structure $\overset{\circ}{\mathbf{F}}$ from (2.13) is introduced. It has the property of homogeneity and $\left(\overset{\circ}{\mathbf{G}}, \overset{\circ}{\mathbf{F}}\right)$ is a metrical almost $(k-1)\, n-$contact structure on $T^k M$. It depend only on the fundamental function $F\left(x, y^{(1)}\right)$ of the Finsler space F^n. The space $\left(\widetilde{T^k M}, \overset{\circ}{\mathbf{G}}, \overset{\circ}{\mathbf{F}}\right)$ is **the geometrical model** of the Finsler space $F^n = \left(M, F\left(x, y^{(1)}\right)\right)$.

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The Sasaki-Matsumoto N-lift of a Finsler metric

Let M be a real n-dimensional C^{∞}-manifold and $\left(T^k M, \pi^k, M\right)$ its tangent bundle of order k (or $k-$jet bundle, or tangent bundle of the higher order accelerations).

Let us consider the Finsler space $F^n = (M, F)$ with the fundamental function $F\left(x, y^{(1)}\right), F : T^1 M \to R$, and the fundamental tensor $g_{ab}\left(x, y^{(1)}\right)$ on $T^1 M$ given by

$$g_{ab}\left(x, y^{(1)}\right) = \frac{1}{2} \frac{\partial^2 F}{\partial y^{(1)a} \partial y^{(1)b}}, \tag{1.1}$$

where $g_{ab}\left(x, y^{(1)}\right)$ is positively defined on $T^1 M$.

The indices $a, b, ...$ run over set $\{1, 2, ..., n\}$ and Einstein convention of summaring is adopted all over this work.

Let $\gamma^a_{bc}\left(x, y^{(1)}\right)$ be the formal Christoffel symbols of the $g_{ab}\left(x, y^{(1)}\right)$, *i.e.* :

$$\gamma^a_{bc}\left(x, y^{(1)}\right) = \frac{1}{2} g^{ad} \left(\frac{\partial g_{bd}}{\partial x^c} + \frac{\partial g_{dc}}{\partial x^b} - \frac{\partial g_{bc}}{\partial x^d}\right). \tag{1.2}$$

Then, the canonical semispray of F^n is given by

$$\frac{d^2x^a}{dt^2} + 2\underset{(1)}{G^a}\left(x, \frac{dx}{dt}\right) = 0, \tag{1.3}$$

where

$$\underset{(1)}{G^a} = \frac{1}{2}\gamma^a_{bc}\left(x, y^{(1)}\right) y^{(1)b} y^{(1)c}. \tag{1.3'}$$

The canonical nonlinear connection (determined only by the function F of the Finsler space F^n) is the Cartan nonlinear connection with the coefficients

$$G^a_b\left(x, y^{(1)}\right) = \frac{\partial \underset{(1)}{G^a}}{\partial y^{(1)b}}. \tag{1.4}$$

Then, on the domain of chart $\left(\pi^k\right)^{-1}(U) \subset T^kM, U \subset M$, we can consider the functions

$$\begin{aligned} F^*\left(x, y^{(1)}, ..., y^{(k)}\right) &= \left(F \circ \pi^k_1\right)\left(x, y^{(1)}, ..., y^{(k)}\right), \\ g^*_{ab}\left(x, y^{(1)}, ..., y^{(k)}\right) &= \left(g_{ab} \circ \pi^k_1\right)\left(x, y^{(1)}, ..., y^{(k)}\right), \\ &\quad \forall\left(x, y^{(1)}, ..., y^{(k)}\right)(U), \end{aligned}$$

where $\pi^k_1 : T^kM \to TM, \pi^k_1\left(x, y^{(1)}, ..., y^{(k)}\right) = \left(x, y^{(1)}\right)$ is the natural projection. For simplicity, F^* and g^*_{ab} will be denote by the same letters F and g_{ab}.

We have

1^0. The canonical nonlinear connection N on $\widetilde{T^kM} = T^kM \setminus \{0\}$ has the dual coefficients

$$\begin{aligned} &\underset{1}{M}{}^a_b = G^a_b, \\ &\underset{2}{M}{}^a_b = \tfrac{1}{2}\left(C\underset{1}{M}{}^a_b + \underset{1}{M}{}^a_c\underset{1}{M}{}^c_b\right), \\ &\dots\dots\dots\dots\dots\dots \\ &\underset{k}{M}{}^a_b = \tfrac{1}{k}\left(C\underset{k-1}{M}{}^a_b + \underset{1}{M}{}^a_c\underset{k-1}{M}{}^c_b\right), \end{aligned} \tag{1.5}$$

where C is the operator

$$C = y^{(1)a}\frac{\partial}{\partial x^a} + 2y^{(2)a}\frac{\partial}{\partial y^{(1)a}} + ... + ky^{(k)a}\frac{\partial}{\partial y^{(k-1)a}} \tag{1.6}$$

2^0. The Liouville d-vector field $z^{(k)}$ corresponding to the canonical nonlinear connection N is given by

$$kz^{(k)a} = ky^{(k)a} + (k-1)\, y^{(k-1)b}\underset{1}{M}{}^a_b + ... + y^{(1)b}\underset{k-1}{M}{}^a_b. \tag{1.7}$$

3^0. The following Lagrangian

$$L\left(x, y^{(1)}, ..., y^{(k)}\right) = g_{ab}\left(x, y^{(1)}\right) z^{(k)a} z^{(k)b}, \tag{1.8}$$

is a regular Lagrangian on $\widetilde{T^kM}$, determined only by $F\left(x, y^{(1)}\right)$ because g_{ab} and $z^{(k)}$ have this property.

4^0. Its fundamental tensor field coincide with the fundamental tensor field on Finsler space F^n, namely on $\widetilde{T^kM}$ we have

$$\frac{1}{2}\frac{\partial^2 L}{\partial z^{(k)a}\partial z^{(k)b}} = g_{ab}\left(x, y^{(1)}\right). \tag{1.9}$$

5^0. N determines the direct decomposition

$$T_uT^kM = N_0(u) \oplus N_1(u) \oplus ... \oplus N_{k-1}(u) \oplus V_k(u), \quad \forall u \in T^kM. \tag{1.10}$$

6^0. The adapted cobasis $\left\{dx^a, \delta y^{(1)a}, ..., \delta y^{(k)a}\right\}$ and the adapted basis $\left\{\frac{\delta}{\delta x^a}, \frac{\delta}{\delta y^{(1)a}}, ..., \frac{\delta}{\delta y^{(k-1)a}}, \frac{\delta}{\delta y^{(k)a}}\right\}$ to N are depending only on fundamental function $F\left(x, y^{(1)}\right)$ of Finsler space F^n, where

$$\begin{aligned}
&\delta y^{(1)a} = dy^{(1)a} + \underset{1}{M}{}^a_c dx^c,\\
&\delta y^{(2)a} = dy^{(2)a} + \underset{1}{M}{}^a_c dy^{(1)c} + \underset{2}{M}{}^a_c dx^c,\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&\delta y^{(k)a} = dy^{(k)a} + \underset{1}{M}{}^a_c dy^{(k-1)c} + ... + \underset{k-1}{M}{}^a_c dy^{(1)c} + \underset{k}{M}{}^a_c dx^c,
\end{aligned} \tag{1.11}$$

and

$$\begin{aligned}
&\frac{\delta}{\delta x^a} = \frac{\partial}{\partial x^a} - \underset{1}{N}{}^c_a\frac{\partial}{\partial y^{(1)c}} - \underset{2}{N}{}^c_a\frac{\partial}{\partial y^{(2)c}} - ... - \underset{k}{N}{}^c_a\frac{\partial}{\partial y^{(k)c}},\\
&\frac{\delta}{\delta y^{(1)a}} = \frac{\partial}{\partial y^{(1)a}} - \underset{1}{N}{}^c_a\frac{\partial}{\partial y^{(2)c}} - ... - \underset{k-1}{N}{}^c_a\frac{\partial}{\partial y^{(k)c}},\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&\frac{\delta}{\delta y^{(k-1)a}} = \frac{\partial}{\partial y^{(k-1)c}} - \underset{1}{N}{}^c_a\frac{\partial}{\partial y^{(k)c}}.
\end{aligned} \tag{1.11'}$$

We know that

$$\begin{aligned}
&\underset{1}{N}{}^a_b = \underset{1}{M}{}^a_b, \underset{2}{N}{}^a_b = \underset{2}{M}{}^a_b - \underset{1}{M}{}^c_b\underset{1}{M}{}^a_c, ...,\\
&\underset{k}{N}{}^a_b = \underset{k}{M}{}^a_b - \underset{1}{M}{}^c_b\underset{k-1}{N}{}^a_c - ... - \underset{k-2}{M}{}^c_b\underset{2}{N}{}^a_c - \underset{k-1}{M}{}^c_b\underset{1}{N}{}^a_c,
\end{aligned} \tag{1.12}$$

and conversely

$$\begin{aligned}
&\underset{1}{M}{}^a_b = \underset{1}{N}{}^a_b, \underset{2}{M}{}^a_b = \underset{2}{N}{}^a_b + \underset{1}{N}{}^a_c\underset{1}{M}{}^c_b, ...,\\
&\underset{k}{M}{}^a_b = \underset{k}{N}{}^a_b + \underset{k-1}{N}{}^a_c\underset{1}{M}{}^c_b + ... + \underset{2}{N}{}^a_c\underset{k-2}{M}{}^c_b + \underset{1}{N}{}^a_c\underset{k-1}{M}{}^c_b.
\end{aligned} \tag{1.12'}$$

Then, the Sasaki-Matsumoto $N-$lift of $g_{ab}\left(x, y^{(1)}\right)$ to T^kM is defined by

$$\mathbf{G}(u) = g_{ab}\left(x, y^{(1)}\right) dx^a \otimes dx^b + \sum_{\beta=1}^{k} g_{ab}\left(x, y^{(1)}\right) \delta y^{(\beta)a} \otimes \delta y^{(\beta)b}, \forall u \in \widetilde{T^kM}. \tag{1.13}$$

The following properties hold:

7^0. $\mathbf{G}$ is globally defined on T^kM.

8^0. $\mathbf{G}$ is a Riemannian structure on T^kM determined only by the Finsler space F^n.

9^0. $\mathbf{G}$ is not homogeneous on the fibres of T^kM.

Namely, for the homothety $h_t : \left(x, y^{(1)}, ..., y^{(k)}\right) \to \left(x, ty^{(1)}, ..., t^k y^{(k)}\right), \forall t \in R_*^+$, we get

$$(G \circ h_t)(u) = g_{ab}\left(x, y^{(1)}\right) dx^a \otimes dx^b + \sum_{\beta=1}^{k} t^{2\beta} g_{ab}\left(x, y^{(1)}\right) \delta y^{(\beta)a} \otimes \delta y^{(\beta)b} \neq G(u).$$

Let us consider the $\mathcal{F}\left(T^k M\right)-$ linear mapping $\mathbf{F} : \chi\left(T^k M\right) \to \chi\left(T^k M\right)$ given in the adapted basis (1.11') by

$$\mathbf{F}\left(\frac{\delta}{\delta x^a}\right) = -\frac{\partial}{\partial y^{(k)a}}, \tag{1.14}$$
$$\mathbf{F}\left(\frac{\delta}{\delta y^{(1)a}}\right) = ... = \mathbf{F}\left(\frac{\delta}{\delta y^{(k-1)a}}\right) = 0,$$
$$\mathbf{F}\frac{\partial}{\partial y^{(k)a}} = \frac{\delta}{\delta x^a}.$$

It follows that:

10^0. $\mathbf{F}$ is globally defined on $T^k M$ and it is a $d-$ tensor field of type $(1,1)$.
11^0. $\mathbf{F}$ is an $(k-1)\,n-$contact structure : $\mathbf{F}^3 + \mathbf{F} = 0$.
12^0. $\mathbf{F}$ depend only on the fundamental function $F\left(x, y^{(1)}\right)$ of Finsler space F^n.
13^0. The pair $(\mathbf{G}, \mathbf{F})$ is a Riemannian almost $(k-1)\,n-$contact structure on $T^k M$:

$$\mathbf{G}(\mathbf{F}X, Y) = -\mathbf{G}(X, \mathbf{F}Y), \forall X, Y \in \chi\left(T^k M\right).$$

Consequently, we get

Theorem 1.1 *The space $\left(T^k M, \mathbf{G}, \mathbf{F}\right)$ is a Riemannian almost $(k-1)\,n-$contact space depending only on the fundamental function $F\left(x, y^{(1)}\right)$ of the Finsler space $F^n = (M, F)$.*

The previous space, called **"the geometrical model on $T^k M$ of the Finsler space"** (M, F) is important in the study of the geometry of the initial Finsler space $F^n = (M, F)$.

The homogeneous prolongation to $T^k M$ of a Finsler metric

We define a new prolongation $\overset{\circ}{G}$ on $T^k M$ of the fundamental tensor field $g_{ab}\left(x, y^{(1)}\right)$ of a Finsler space $F^n = (M, F)$, which satisfies the following conditions:

1^0. $\overset{\circ}{G}$ is $0-$ homogeneous with respect to $y^{(1)a}, y^{(2)a}, ...,$and $y^{(k)a}$.
2^0. It depends only on the fundamental function $F\left(x, y^{(1)}\right)$.
3^0. In the mechanical meaning the terms of $\overset{\circ}{G}$ have the same physical dimensions.

Definition 2.1. *We call the homogeneous prolongation to $T^k M$ of the fundamental tensor field $g_{ab}\left(x, y^{(1)}\right)$ of a Finsler space $F^n = (M, F)$,the following tensor field on $T^k M$:*

$$\overset{\circ}{G}(u) = g_{ab}\left(x, y^{(1)}\right) dx^a \otimes dy^b + \sum_{\beta=1}^{k} \frac{\mathbf{a}^{2\beta}}{\|y^{(1)}\|^{2\beta}} g_{ab}\left(x, y^{(1)}\right) \delta y^{(\beta)a} \otimes \delta y^{(\beta)b}, \forall u \in \widetilde{T^k M}, \tag{2.1}$$

where $\mathbf{a} > \mathbf{0}$ *is a constant imposed by application in order to preserve the physical dimension of the components of* $\overset{\circ}{G}$, *and* $\left\|y^{(1)}\right\|^2$ *is the square of the norm of the first Liouvill vector field*

$$\left\|y^{(1)}\right\|^2 = g_{ab}\left(x, y^{(1)}\right) y^{(1)a} y^{(1)b}. \tag{2.2}$$

We get, without difficulties:

Theorem 2.1. 1.*The pair* $\left(\widetilde{T^kM}, \overset{\circ}{G}\right)$ *is a Riemann space.*

2. $\overset{\circ}{G}$ *is a* $0-$ *homogeneous tensor field with respect to* $y^{(\beta)a}, (\beta = 1, ..., k)$.

3. $\overset{\circ}{G}$ *depends only on the fundamental function* $F\left(x, y^{(1)}\right)$ *of the Finsler space* F^n.

4. *The distributions* $N_0, N_1, ..., N_{k-1}$ *and* V_k *are orthogonal, in pairs, with respect to* $\overset{\circ}{G}$.

We can write $\overset{\circ}{G}$ in the form

$$\overset{\circ}{\mathbf{G}} = \overset{\circ}{\mathbf{G}}{}^H + \overset{\circ}{\mathbf{G}}{}^{V_1} + ... + \overset{\circ}{\mathbf{G}}{}^{V_k}, \tag{2.3}$$

where

$$\overset{\circ}{\mathbf{G}}{}^H = g_{ab}\left(x, y^{(1)}\right) dx^a \otimes dx^b, \overset{\circ}{\mathbf{G}}{}^{V_\beta} = g_{ab}\left(x, y^{(1)}\right) dy^{(\beta)a} \otimes dy^{(\beta)b} \tag{2.4}$$

and

$$\underset{(\beta)}{g}{}_{ab}\left(x, y^{(1)}\right) = \frac{\mathbf{a}^{2\beta}}{\left\|y^{(1)}\right\|^{2\beta}} g_{ab}\left(x, y^{(1)}\right), \quad (\beta = 1, ..., k). \tag{2.5}$$

As usually, let us denote

$$\partial_a = \frac{\partial}{\partial x^a}, \dot{\partial}_{1a} = \frac{\partial}{\partial y^{(1)a}}, ..., \dot{\partial}_{ka} = \frac{\partial}{\partial y^{(k)a}},$$

and from now on we denote the adapted basis (1.11') by

$$\left\{\delta_a, \delta_{1a}, ..., \delta_{(k-1)a}, \delta_{ka}\right\}.$$

In order to study the geometry of Riemann space $\left(\widetilde{T^kM}, \overset{\circ}{G}\right)$, we can apply the theory of the $(h, v_1, ..., v_k)-$ Riemannian metric given by author in [5] (for $k = 2$, see [2], [4]).

A linear connection D on T^kM is called a metrical $N-$linear connection with respect to $\overset{\circ}{\mathbf{G}}$ if $D_X\overset{\circ}{\mathbf{G}} = 0, \forall X \in \chi\left(T^kM\right)$ and it preserves by paralelism the horizontal and vertical distributions $N_0, N_1, ..., N_{k-1},\ V_k$.

We can easily prove the existence of the metrical $N-$linear connections in the adapted basis. To this aim we represent a linear connection D in the adapted basis in the following form:

$$\begin{gathered}
D_{\delta_c}\delta_b = \underset{(00)}{\overset{0}{L}}{}^a{}_{bc}\delta_a + \sum_{\beta=1}^{k}\underset{(00)}{\overset{\beta}{L}}{}^a{}_{bc}\delta_{\beta a}, \\
D_{\delta_c}\delta_{\gamma b} = \underset{(\gamma 0)}{\overset{0}{L}}{}^a{}_{bc}\delta_a + \sum_{\beta=1}^{k}\underset{(\gamma 0)}{\overset{\beta}{L}}{}^a{}_{bc}\delta_{\beta a}, \quad \left(\gamma = 1, ..., k; \delta_{ka} = \dot{\partial}_{ka}\right), \\
D_{\delta_{1c}}\delta_b = \underset{(01)}{\overset{0}{C}}{}^a{}_{bc}\delta_a + \sum_{\beta=1}^{k}\underset{(01)}{\overset{\beta}{C}}{}^a{}_{bc}\delta_{\beta a}, \\
D_{\delta_{1c}}\delta_{\gamma b} = \underset{(\gamma 1)}{\overset{0}{C}}{}^a{}_{bc}\delta_a + \sum_{\beta=1}^{k}\underset{(\gamma 1)}{\overset{\beta}{C}}{}^a{}_{bc}\delta_{\beta a}, \quad \left(\gamma = 1, ..., k; \delta_{ka} = \dot{\partial}_{ka}\right), \\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
D_{\delta_{kc}}\delta_b = \underset{(0k)}{\overset{0}{C}}{}^a{}_{bc}\delta_a + \sum_{\beta=1}^{k}\underset{(0k)}{\overset{\beta}{C}}{}^a{}_{bc}\delta_{\beta a}, \\
D_{\delta_{kc}}\delta_{\gamma b} = \underset{(\gamma k)}{\overset{0}{C}}{}^a{}_{bc}\delta_a + \sum_{\beta=1}^{k}\underset{(\gamma k)}{\overset{\beta}{C}}{}^a{}_{bc}\delta_{\beta a}, \quad \left(\gamma = 1, ..., k; \delta_{ka} = \dot{\partial}_{ka}\right).
\end{gathered} \tag{2.6}$$

The system of functions

$$\left(\underset{(00)}{\overset{\alpha}{L}}{}^{a}{}_{bc},\ \underset{(\beta 0)}{\overset{\alpha}{L}}{}^{a}{}_{bc},\ \underset{(01)}{\overset{\alpha}{C}}{}^{a}{}_{bc},\ \underset{(\beta 1)}{\overset{\alpha}{C}}{}^{a}{}_{bc},\ldots,\ \underset{(0k)}{\overset{\alpha}{C}}{}^{a}{}_{bc},\ \underset{(\beta k)}{\overset{\alpha}{C}}{}^{a}{}_{bc}\right),\quad (\alpha=0,1,\ldots,k;\quad \beta=1,\ldots,k),$$

are the coefficients of D and

$$\left(\underset{(00)}{\overset{0}{L}}{}^{a}{}_{bc},\ \underset{(\beta 0)}{\overset{\beta}{L}}{}^{a}{}_{bc},\ \underset{(01)}{\overset{0}{C}}{}^{a}{}_{bc},\ \underset{(\beta 1)}{\overset{\beta}{C}}{}^{a}{}_{bc},\ldots,\ \underset{(0k)}{\overset{0}{C}}{}^{a}{}_{bc},\ \underset{(\beta k)}{\overset{\beta}{C}}{}^{a}{}_{bc}\right),\quad (\beta=1,\ldots,k),$$

are the coefficients of an $N-$linear connection $D\Gamma(N)$ on T^kM.

Also, we will denote the coefficients of $D\Gamma(N)$ with

$$\left(\underset{(00)}{\overset{H}{L}}{}^{a}{}_{bc},\ \underset{(\beta 0)}{\overset{V_\beta}{L}}{}^{a}{}_{bc},\ \underset{(01)}{\overset{H}{C}}{}^{a}{}_{bc},\ \underset{(\beta 1)}{\overset{V_\beta}{C}}{}^{a}{}_{bc},\ldots,\ \underset{(0k)}{\overset{H}{C}}{}^{a}{}_{bc},\ \underset{(\beta k)}{\overset{V_\beta}{C}}{}^{a}{}_{bc}\right),\quad (\beta=1,\ldots,k).$$

It is not difficult to prove

Theorem 2.2. *There exist metrical $N-$linear connection $D\Gamma(N)$ on T^kM with respect to the homogeneous prolongation $\overset{\circ}{\mathbf{G}}$, which depend only of the fundamental function $F\left(x,y^{(1)}\right)$ of the Finsler space F^n. One of them has the* **"horizontal"** *coefficients:*

$$\begin{aligned}\underset{(00)}{\overset{H}{L}}{}^{a}{}_{bc} &= \frac{1}{2}g^{ad}\left(\delta_b g_{dc}+\delta_c g_{bd}-\delta_d g_{bc}\right),\\ \underset{(\beta 0)}{\overset{V_\beta}{L}}{}^{a}{}_{bc} &= \frac{1}{2}\underset{(\beta)}{g}{}^{ad}\left(\delta_b \underset{(\beta)}{g}{}_{dc}+\delta_c \underset{(\beta)}{g}{}_{bd}-\delta_d \underset{(\beta)}{g}{}_{bc}\right),\quad (\beta=1,\ldots,k),\end{aligned}\tag{2.7}$$

the **"$\mathbf{v}_1-$vertical"** *coefficients:*

$$\begin{aligned}\underset{(\beta 1)}{\overset{V_\beta}{C}}{}^{a}{}_{bc} &= \frac{1}{2}g^{ad}\left(\delta_{1b} g_{dc}+\delta_{1c} g_{bd}-\delta_{1d} g_{bc}\right),\\ \underset{(\beta 1)}{\overset{V_\beta}{C}}{}^{a}{}_{bc} &= \frac{1}{2}\underset{(\beta)}{g}{}^{ad}\left(\delta_{1b} \underset{(\beta)}{g}{}_{dc}+\delta_{1c} \underset{(\beta)}{g}{}_{bd}-\delta_{1d} \underset{(\beta)}{g}{}_{bc}\right),\quad (\beta=1,\ldots,k),\end{aligned}\tag{20}$$

and the **"$\mathbf{v}_\gamma-$vertical"** *coefficients vanish:*

$$\underset{(0\gamma)}{\overset{H}{C}}{}^{a}{}_{bc}=\underset{(1\gamma)}{\overset{V_1}{C}}{}^{a}{}_{bc}=\ldots=\underset{(k\gamma)}{\overset{V_k}{C}}{}^{a}{}_{bc}=0,\quad (\gamma=2,\ldots,k).\tag{2.9}$$

Let us remark the particular form of the metrical $N-$linear connection $D\Gamma(N)$ in (2.7), (2.8) and (2.9). Because it depends only on the fundamental function $F\left(x,y^{(1)}\right)$ *of the Finsler space F^n*, $D\Gamma(N)$ from the Theorem 2.2 will be called the **canonical** metrical $N-$linear connection of the space $\left(\widetilde{T^kM},\overset{\circ}{\mathbf{G}}\right)$.

Let us denote

$$\sigma_c=-\frac{1}{2}\frac{1}{F^2}\delta_c F^2,\ \tau_c=-\frac{1}{2}\frac{1}{F^2}\delta_{1c}F^2.\tag{2.10}$$

We obtain

Theorem 2.3. *The coeffients of the metrical $N-$linear connection $D\Gamma(N)$ with respect to $\overset{\circ}{\mathbf{G}}$ given by* (2.1), *satisfy the following equations:*

$$\begin{aligned} &\underset{(\beta 0)}{\overset{V_\beta}{C}}{}^a{}_{bc} = \underset{(00)}{\overset{H}{L}}{}^a{}_{bc} + \beta\left(\delta^a_b\sigma_c + \delta^a_c\sigma_b - g_{bc}g^{ad}\sigma_\alpha\right), \\ &\underset{(\beta 1)}{\overset{V_\beta}{C}}{}^a{}_{bc} = \underset{(01)}{\overset{V_\beta}{C}}{}^a{}_{bc} + \beta\left(\delta^a_b\sigma_c + \delta^a_c\sigma_b - g_{bc}g^{ad}\sigma_\alpha\right), \quad (\beta = 1, ..., k). \end{aligned} \tag{2.11}$$

Indeed, substituting the tensors $\underset{(1)}{g}_{ab}, \underset{(2)}{g}_{ab}, ..., \underset{(k)}{g}_{ab}$ given by (2.5) in (2.7) and (2.8) and using (2.10), one obtains (2.11).

It is not difficult to prove

Theorem 2.4. *The coefficients of the canonical metrical $N-$linear connection* $D\Gamma(N) = \left(\underset{(00)}{\overset{H}{L}}{}^a{}_{bc}, \underset{(\beta 0)}{\overset{V_\beta}{L}}{}^a{}_{bc}, \underset{(01)}{\overset{H}{C}}{}^a{}_{bc}, \underset{(\beta 1)}{\overset{V_\beta}{C}}{}^a{}_{bc}, ..., \underset{(0k)}{\overset{H}{C}}{}^a{}_{bc}, \underset{(\beta k)}{\overset{V_\beta}{C}}{}^a{}_{bc}\right)$, $(\beta = 1, ..., k)$ *satisfy the equations*

$$\begin{aligned} &\underset{(10)}{\overset{V_1}{L}}{}^a{}_{bc} = \underset{(00)}{\overset{H}{L}}{}^a{}_{bc} + \delta^a_b\sigma_c + \delta^a_c\sigma_b - g_{bc}g^{ad}\sigma_\alpha, \\ &\underset{(20)}{\overset{V_2}{L}}{}^a{}_{bc} = \underset{(10)}{\overset{V_2}{L}}{}^a{}_{bc} + \delta^a_b\sigma_c + \delta^a_c\sigma_b - g_{bc}g^{ad}\sigma_\alpha, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\underset{(k0)}{\overset{V_k}{L}}{}^a{}_{bc} = \underset{(k-10)}{\overset{V_{k-1}}{L}}{}^a{}_{bc} + \delta^a_b\sigma_c + \delta^a_c\sigma_b - g_{bc}g^{ad}\sigma_\alpha, \\ &\underset{(11)}{\overset{V_1}{C}}{}^a{}_{bc} = \underset{(01)}{\overset{H}{C}}{}^a{}_{bc} + \delta^a_b\sigma_c + \delta^a_c\sigma_b - g_{bc}g^{ad}\sigma_\alpha, \\ &\underset{(21)}{\overset{V_2}{C}}{}^a{}_{bc} = \underset{(11)}{\overset{V_1}{C}}{}^a{}_{bc} + \delta^a_b\sigma_c + \delta^a_c\sigma_b - g_{bc}g^{ad}\sigma_\alpha, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\underset{(k1)}{\overset{V_k}{C}}{}^a{}_{bc} = \underset{(k-11)}{\overset{V_{k-1}}{C}}{}^a{}_{bc} + \delta^a_b\sigma_c + \delta^a_c\sigma_b - g_{bc}g^{ad}\sigma_\alpha, \\ &\underset{(0\gamma)}{\overset{H}{C}}{}^a{}_{bc} = \underset{(1\gamma)}{\overset{V_1}{C}}{}^a{}_{bc} = ... = \underset{(k\gamma)}{\overset{V_k}{C}}{}^a{}_{bc} = 0, \quad (\gamma = 2, ..., k). \end{aligned} \tag{2.12}$$

The particular form (2.12) of the canonical metrical $N-$linear connection shows that the curvature of the v_k-connection $\left(\underset{(k0)}{\overset{V_k}{L}}{}^a{}_{bc}, \underset{(k1)}{\overset{V_k}{C}}{}^a{}_{bc}, ..., \underset{(kk)}{\overset{V_k}{C}}{}^a{}_{bc}\right)$ lead to the Weyl's conformal curvature tensor with respect to the curvature of the $v_{k-1}-$connection $\left(\underset{(k-10)}{\overset{V_{k-1}}{L}}{}^a{}_{bc}, \underset{(k-11)}{\overset{Vk_{-1}}{C}}{}^a{}_{bc}, ..., \underset{(k-1k)}{\overset{Vk-1}{C}}{}^a{}_{bc}\right)$, ...,and the curvature of the v_1-connection $\left(\underset{(10)}{\overset{V_1}{L}}{}^a{}_{bc}, \underset{(11)}{\overset{V_1}{C}}{}^a{}_{bc}, ..., \underset{(1k)}{\overset{V_2}{C}}{}^a{}_{bc}\right)$ lead to the Weyl's conformal curvature of the $h-$connection $\left(\underset{(00)}{\overset{H}{L}}{}^a{}_{bc}, \underset{(01)}{\overset{H}{C}}{}^a{}_{bc}, ..., \underset{(0k)}{\overset{H}{C}}{}^a{}_{bc}\right)$.

This property shows the necessity to construct a gauge theory in the Asanov sense, [1], for the Riemannian metric given on $\widetilde{T^kM}$ by the prolongation $\overset{\circ}{\mathbf{G}}$, from (4.1).

Now, we remark that the almost $(k-1)n-$contact structure $\mathbf{F}$ defined in (1.14) has not the property of homogeneity. The $\mathcal{F}\left(\widetilde{T^kM}\right)-$linear mapping $\mathbf{F} : \chi\left(\widetilde{T^kM}\right) \to \chi\left(\widetilde{T^kM}\right)$, applies the $1-$homogeneous vector field δ_a into the $(1-k)-$homogeneous vector field $\delta_{ka} = \dot{\partial}_{ka}, (a = 1, ..., n)$.

Therefore, we consider the $\mathcal{F}\left(\widetilde{T^kM}\right)$ −linear mapping $\overset{\circ}{\mathbf{F}} : \chi\left(\widetilde{T^kM}\right) \to \chi\left(\widetilde{T^kM}\right)$, given in the adapted basis by

$$\begin{aligned} \overset{\circ}{\mathbf{F}}(\delta_a) &= -\frac{\left\|y^{(1)}\right\|^k}{\mathbf{a}^k}\dot{\partial}_{ka}, \\ \overset{\circ}{\mathbf{F}}(\delta_{1a}) = \ldots = \overset{\circ}{\mathbf{F}}(\delta_{k-1a}) &= 0, \\ \overset{\circ}{\mathbf{F}}\left(\dot{\partial}_{ka}\right) &= \frac{\mathbf{a}^k}{\left\|y^{(1)}\right\|^k}\delta_a. \end{aligned} \tag{2.13}$$

By direct calculus, we can prove:

Theorem 2.5. $\overset{\circ}{\mathbf{F}}$ *has the following properties:*

1. $\overset{\circ}{\mathbf{F}}$ *is a tensor field of type* (1.1) *on* $\left(\widetilde{T^kM}\right)$.

2. $\overset{\circ}{\mathbf{F}}$ *is an almost* $(k-1)\,n-$*contact structure on* $\widetilde{T^kM} : \mathbf{F}^3 + \mathbf{F} = 0$.

3. $\overset{\circ}{\mathbf{F}}$ *depends only the fundamental function* $F\left(x, y^{(1)}\right)$ *of the Finsler space* F^n.

4. $\overset{\circ}{\mathbf{F}}$ *is homogeneous on the fibres on* $\widetilde{T^kM}$.

5. *The pair* $\left(\overset{\circ}{\mathbf{G}}, \overset{\circ}{\mathbf{F}}\right)$ *is a metrical* $(k-1)\,n-$*contact structure on* $\widetilde{T^kM}$:

$$\overset{\circ}{\mathbf{G}}\left(\overset{\circ}{\mathbf{F}}X, Y\right) = -\overset{\circ}{\mathbf{G}}\left(X, \overset{\circ}{\mathbf{F}}Y\right), \quad \forall X, Y \in \chi\left(\widetilde{T^kM}\right).$$

The space $\left(\widetilde{T^kM}, \overset{\circ}{\mathbf{G}}, \overset{\circ}{\mathbf{F}}\right)$ is **the geometrical model** of the Finsler space $F^n = (M, F)$, with respect to the homogeneous lift $\overset{\circ}{\mathbf{G}}$ given by (2.1). It can be used for studying the Finslerian higher order gauge theory and, in general, the geometry of the Finsler space $F^n = (M, F)$.

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