GENERALIZATION OF SCALAR PRODUCT AXIOMS

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The concept of scalar product is vital in studying basic properties of either Euclidean or pseudo-Euclidean spaces. A generalizing of a special sub-class of Finslerian spaces, that we will call the polylinear, is presented in the work. The idea of scalar polyproduct and of related fundamental metric polyform has been introduced axiomatically. The definition of different metric parameters such as the vector length and the angle between vectors are founded on the idea. The concept of orthogonallity is also generalized. Some peculiarities of the geometry of the four-dimensional linear Finslerian space related to the algebra of commutative-associative hypercomplex numbers, that are called Quadranumerical, are proved in the concrete polyform.

1. The scalar product of the Euclidean spaces

For the last two thousand years that have past since the appearance of the famous "Beginnings" mathematics have tried a number of methods of describing the Euclidean spaces. The axiom systems by Euclid and Gilbert are the best well-known ones. But taking into consideration the modern attitude, the system of axioms that uses the ideas of the real number, the linear space, and the scalar product [1] is considered to be the most convenient. At the same time a few know that the latter case owes its appearance in geometry to a discovery of the non-commutative algebra of four-component hyper-complex numbers discovered in 1843 by William Hamilton, he called it the algebra of quaternions [2]. The discovery was preceded by several years of attempts to find three-component numbers, the triplets, that could be confronted to the vectors of the common space the same way as the complex numbers are confronted to the vectors of the Euclidean Plane. The solution was found when Hamilton rejected the commutative multiplication and in place of the triplets limited himself to the four-component numbers.

By definition a quaternion is a hyper-complex number, that can be presented as a linear combination:

$$X = x_0 + i \cdot x_1 + j \cdot x_2 + k \cdot x_3,$$

where x_i are real numbers, and i, j, k are pair-wisely different imaginary units, so that $i^2 = j^2 = k^2 = -1$ and $ij + ji = jk + kj = ki + ik = 0$. These rules including the rule of multiplication on the common real unit, sometimes are set into the so called table of multiplication of hypercomplex numbers, that in the case of quaternions looks the following way:

	1	i	j	k
1	1	i	j	k
i	i	-1	k	$-j$
j	j	$-k$	-1	i
k	k	j	$-i$	-1

Hamilton suggested that in the quaternion we should distinguish the scalar part x_0 from the vector part $\mathbf{V}_x = \mathbf{i} \cdot x_1 + \mathbf{j} \cdot x_2 + \mathbf{k} \cdot x_3$. In this case, as it is easy to check, the product of 2 vector quaternions is a common quaternion:

$$\mathbf{V}_x \mathbf{V}_y = (-x_1 y_1 - x_2 y_2 - x_3 y_3) + [\mathbf{i}(x_2 y_3 - x_3 y_2) + \mathbf{j}(x_3 y_1 - x_1 y_3) + \mathbf{k}(x_1 y_2 - x_2 y_1)],$$

whose scalar part has a symmetric bilinear form, and the vector part looks like a conventional vector multiplication. As a matter of fact, the term of scalar and vector product appeared right from here, and for the first time were introduced by Hamilton.

The first explorers of the quaternions were looking at them mainly as at an opportunity of using algebraic methods while operating with points and vectors of common space, though it is more natural to correspond these hyper-complex numbers with the four-dimensional space. Hamilton himself knew about this, he thought that this circumstance once would be used to describe the time. In this case quaternions would become a natural instrument not only in geometry, but also in physics.

Unfortunately, nowadays only some specialists know quaternions. It is explained by the fact that the idea of scalar product that originates from the quaternion algebra was very convenient and soon became an independent geometrical category, and practically stamped the hyper-complex numbers that had given birth to it. There began a debate among physics and mathematicians between the adherents of the quaternion algebra and of the arising vector calculus. As is well-known, the vector approach won, this fact to a certain extent owes to objective difficulties of quaternion diffusion into algebra and the function of the complex variable, that is conditioned to the peculiarities of non-commutative multiplication.

The scalar product that is connected with the quaternion can be applied only to the three-dimensional vectors. But if we separate the idea of scalar product from concrete numbers and generalize it to the field of arbitrary dimensionality, the advantages of the concept (the opportunity to define the length of vectors and angles between them mathematically) will still be preserved. For this we should postulate a symmetrical bilinear form of two vectors $(\mathbf{A}, \mathbf{B}) = \alpha_{ij} a_i b_j$ in the affine m-dimensional space. Reciprocally corresponding quadratic form $(\mathbf{A}, \mathbf{A})$ must be not negative. Then by definition we accept that the affine map that maps the vector $\mathbf{A}$ onto $\mathbf{A}'$ is congruent if it leaves the form invariant:

$$(\mathbf{A}, \mathbf{A}) = (\mathbf{A}', \mathbf{A}').$$

Two figures that can be mapped one onto another by a congruent reflection are congruent. By this fact the idea of congruence is defined in the axiomatic construction of the Euclidean geometry. For a congruent map takes place not only invariance of the quadratic form but also the invariance o the bilinear form:

$$(\mathbf{A}, \mathbf{B}) = (\mathbf{A}', \mathbf{B}').$$

For the vectors $\mathbf{A}$ and $\mathbf{A}'$ are congruent if and only if:

$$(\mathbf{A}, \mathbf{A}) = (\mathbf{A}', \mathbf{A}'),$$

it is possible to introduce the $(\mathbf{A}, \mathbf{A})$ as a numerical characteristic of the vector $\mathbf{A}$. But still it is more traditional to use the value of the positive square root of $(\mathbf{A}, \mathbf{A})$, that by definition is called the length of the vector $\mathbf{A}$ and usually is defined as

$$|\mathbf{A}| = (\mathbf{A}, \mathbf{A})^{1/2}.$$

Such definition lets us introduce the definition of the unit vector. Its relationship with common vectors is revealed in the following relation:

$$\mathbf{a} = \mathbf{A}/|\mathbf{A}|.$$

If $\mathbf{a}$ and $\mathbf{b}$, and $\mathbf{a}'$ and $\mathbf{b}'$, are two pairs of unit length vectors, then the figure, built by the first two vectors, is congruent to the figure, constructed by the two latter ones, only when the equality

$$(\mathbf{a}, \mathbf{b}) = (\mathbf{a}', \mathbf{b}')$$

is held true. The angle is considered to be the representative of congruency in the Euclidean spaces. But the mere numerical characteristic is related not to bilinear form of unit vectors, but to transcendental function of its inverse cosine

$$\phi = \arccos(\mathbf{a}, \mathbf{b}).$$

This definition of the angle is equivalent to the statement that the length of the arc on the unit sphere between the ends of the vectors $\mathbf{a}$ and $\mathbf{b}$ is the angle. Such complication of the numerical angle measure is compensated by the obtained property of additivity. When composing two angles laying on the same plane their value is summed up.

The property of perpendicularity of directions is a particular consequence of the idea of the angle. The perpendicular condition of two vectors consists in equality to 0 of the value of their bilinear form. The particular status of the perpendicular directions is accounted for many reasons, for example, for example by the simplification of the form of the quadratic metric function, presented in the basis all vectors of which are reciprocally perpendicular.

Two-dimensional case stands out among all the Euclidean spaces with quadratic metric function. This peculiarity is reflected in the Liouville theorem, that proves that in the three- or more-dimensional Euclidean (or pseudo-Euclidean) spaces the conformal transformations are limited to inversions, dilations, translations and rotations [3]. In other words, there are essentially more transformations that are related to conformal in the two-dimensional case. Mathematically this fact is reflected in the vast majority of analytical functions of the complex variable. To each of them a certain conformal reflection of the Euclidean plane is related.

2. The scalar product of the pseudo-Euclidean spaces

It is well-known that if a symmetrical bilinear form postulated over the affine space creates an alternating-sign quadratic form, then the geometry assigned by it becomes being of not Euclidean but Pseudo-Euclidean type [4]. We can unify both types of geometries by surrendering the claim about the positivity of the quadratic form. This unified system, in particular, can be presented with the following set:

(a): every 2 vectors $\mathbf{A}$ and $\mathbf{B}$ of the linear space are associated with a certain real number labeled by

$$k = (\mathbf{A}, \mathbf{B})$$

and called (as well as in the Euclidean case) the scalar product of these vectors;

(b) the scalar product is commutative regarding the permutation of vectors

$$(\mathbf{A}, \mathbf{B}) = (\mathbf{B}, \mathbf{A});$$

(c) the scalar product is distributive regarding the composition of vectors

$$(\mathbf{A} + \mathbf{C}, \mathbf{B}) = (\mathbf{A}, \mathbf{B}) + (\mathbf{C}, \mathbf{B});$$

(d) the real multiplier can be isolated from the scalar product

$$(k\mathbf{A}, \mathbf{B}) = k(\mathbf{A}, \mathbf{B}).$$

The methods of defining the metric characteristics of pseudo-Euclidean spaces, which are the generalizing of the corresponding Euclidean parameters, do not change considerably, that enables us to save their names. So, transformations that leave the quadratic form moduli of all the vectors invariant are of congruent nature:

$$|(\mathbf{A}, \mathbf{A})| = |(\mathbf{A}', \mathbf{A})'|.$$

The vector length is defined as a positive value of the square root of the moduli of the quadratic form:

$$|\mathbf{A}| = |(\mathbf{A}, \mathbf{A})|^{1/2}.$$

But in this case there appear the so called isotropic and imaginary vectors. In the first case the length equals 0 even at non-zero components, and in the second case the quadratic form is negative. The angle between the two directions, as well as in the Euclidean case, is defined by congruence of the figure formed by two unit vectors, and by definition is treated as equal to the special function of their bilinear form:

$$\phi = \operatorname{arcch}(\mathbf{a}, \mathbf{b}),$$

which ensures the additivity of the parameter under plane rotations. So, the angle equals the arc length between a pair of points on the unit sphere. But now, when calculating the angle, it is important to take into consideration the area in which the driving vector that is relative to the isotropic cone is lying, as the indicatrix stops being simply connected.

Also the perpendicular property of vectors is generalized in the pseudo-Euclidean spaces. In this case their scalar product must equal 0. It is customary to call such vectors orthogonal.

The pseudo-Euclidean spaces also admit the generalizing of the idea of a congruent reflection, which is defined as a transformation that saves the similarity of infinitesimal forms. Let us note that, as well as in the Euclidean case, the two-dimensional case, where conformal maps are wider than in higher dimensions, is distinguished in the pseudo-Euclidean space. Let us note another coincidence: The pseudo-Euclidean plane, as well as the Euclidean one, has an algebraic analogue called *double numbers* which differ from the complex by the fact that their square equals not -1, but +1. Such numbers along with the complex ones admit the idea of analytical functions where a correspondence of a conformal reflection of the pseudo-Euclidean plane [5] to each of them can be established. These peculiarities of two-dimensional spaces demonstrate the relationship between the geometries and commutative-associative algebras, for example, the algebras of complex and double numbers.

Apart from the pseudo-Euclidean case other approaches towards generalizing of the conception of the scalar product are known in geometry. The system of axioms for the so called unitary, where the metric function is set in the field of complex and not real numbers, and symplectic spaces where antisymmetric bilinear form [4, 6] is postulated in place of the symmetric, — are sequent to the scalar product.

Analyzing above examined examples of the usage of the concept of scalar product and its generalizing we can note that they are unified by connection with one or another bilinear form. But such form is just a special case of the polylinear form. Then there emerges a question whether it is possible to obtain a substantial geometry if we postulate the three-, four-, and so on up to polylinear symmetric form in place of the bilinear one?

3. The scalar polyproduct

Let us try to preserve all the axioms of the real number and m-dimensional affine spaces as the basis and add the following:

(a): to every of n vectors $\mathbf{A}, \mathbf{B}, \mathbf{C}, \ldots, \mathbf{Z}$ we will associate a real number denoted by

$$k = (\mathbf{A}, \mathbf{B}, \mathbf{C}, \ldots, \mathbf{Z}),$$

which we will call the scalar polyproduct;

(b): let us try to make it the way that the scalar polyproduct would be commutative with respect to permutation of any including vectors

$$(\mathbf{A}, \mathbf{B}, \mathbf{C}, \ldots, \mathbf{Z}) = (\mathbf{B}, \mathbf{A}, \mathbf{C}, \ldots, \mathbf{Z}) = (\mathbf{C}, \mathbf{B}, \mathbf{A}, \ldots, \mathbf{Z}) = \cdots = (\mathbf{Z}, \mathbf{C}, \mathbf{B}, \ldots, \mathbf{A});$$

(c): distributive to their composing

$$(\mathbf{A}, \mathbf{B}, \mathbf{C} + \mathbf{E}, \ldots, \mathbf{Z}) = (\mathbf{A}, \mathbf{B}, \mathbf{C}, \ldots, \mathbf{Z}) + (\mathbf{A}, \mathbf{B}, \mathbf{E}, \ldots, \mathbf{Z});$$

(d): a real multiplier at any vector could be taken outside the scalar polyproduct:

$$(k\mathbf{A}, \mathbf{B}, \mathbf{C}, \ldots, \mathbf{Z}) = k(\mathbf{A}, \mathbf{B}, \mathbf{C}, \ldots, \mathbf{Z}).$$

These axioms just in a way differ from the corresponding axioms of the scalar product. Besides they can be unified into a concept of the symmetric polylinear form, and that is why we will call the space, endowed with one of the forms, *polylinear*. The above examined Euclidean and pseudo-Euclidean spaces, according to their primary definitions, are special cases of the polylinear spaces, in other words they comply to the above given axiom system when $n = 2$, that enables us to call them *bilinear*.

We will call the scalar polyproduct of the same vector, $\mathbf{A}, \mathbf{A}, \ldots, \mathbf{A}$, by analogy with the quadratic form of the bilinear spaces, the *fundamental metric form* of the polylinear space, or simply *n-polyform* of the vector $\mathbf{A}$.

We will call the affine reflections of the polylinear space, that shift the vectors $\mathbf{A}$ into $\mathbf{A}'$, the *congruent* if they leave he moduli of the fundamental metric form invariant:

$$|(\mathbf{A}, \mathbf{A}, \mathbf{A}, \ldots, \mathbf{A})| = |(\mathbf{A}', \mathbf{A}', \mathbf{A}', \ldots, \mathbf{A}')|. \quad (1)$$

It is in our axiomatic construction of the polylinear space where the idea of congruence, and then of other metric notions, will be defined.

If there is a set of objects over which the axioms of the affine space are held true, we can choose any symmetric polylinear form in it and, therefore, the unambiguously connected n-polyform, and "assign" make the latter to be the fundamental metric form and on its basis define the conception of congruence as it has been done above. Then we a metrics gets introduced into the affine space with the help of the form, and it becomes a correct metric geometry. Such construction is not related neither to number of dimensions in the space nor to the specific number of dimensions in the fundamental form, nor with the type of the latter case.

It follows from the properties of the symmetry and from the linearity of the form $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \ldots, \mathbf{Z})$ where correlations, that are more general than (1), are held true for the congruent reflection of the polylinear space:

$$\begin{gathered}(\mathbf{A}, \mathbf{A}, \ldots, \mathbf{A}, \mathbf{B}) = (\mathbf{A}', \mathbf{A}', \ldots, \mathbf{A}', \mathbf{B}'),\\ (\mathbf{A}, \mathbf{A}, \ldots, \mathbf{B}, \mathbf{B}) = (\mathbf{A}', \mathbf{A}', \ldots, \mathbf{B}', \mathbf{B}'),\\ \cdots\cdots\cdots\\ (\mathbf{A}, \mathbf{B}, \ldots, \mathbf{C}, \mathbf{Z}) = (\mathbf{A}', \mathbf{B}', \ldots, \mathbf{C}', \mathbf{Z}').\end{gathered}$$

In other words the congruent reflections of the polylinear spaces leave the polyforms invariant where the vectors are present in different combinations.

We will say that the two vectors of the polylinear space $\mathbf{A}$ and $\mathbf{A}'$ are congruent if the moduli of the corresponding n-polyforms are equal and are nonzero:

$$|(\mathbf{A}, \mathbf{A}, \ldots, \mathbf{A}, \mathbf{A})| = |(\mathbf{A}', \mathbf{A}', \ldots, \mathbf{A}', \mathbf{A}')| \neq 0.$$

By definition it is possible to regard a n-polyform as a numerical parameter of the vector $\mathbf{A}$. But in place of this, as well as in the bilinear spaces, striving for additivity and unambiguity of the properties, we will use the positive root of the n-degree of the absolute value $(\mathbf{A}, \mathbf{A}, \ldots, \mathbf{A}, \mathbf{A})$, calling it the vector length $\mathbf{A}$:

$$|\mathbf{A}| = |(\mathbf{A}, \mathbf{A}, \ldots, \mathbf{A}, \mathbf{A})|^{1/n}.$$

Then the length of the sum of two codirected vectors equals the sum of their length. It is worth noting that this is not the only way of introducing the idea of length with additive properties, but in this approach the length is defined for the maximum number of directions coming from the affine space.

Now it becomes clear to which type of space we should relate the ones we try to construct with the help of the given above axioms or the scalar polyproduct. Firstly, these spaces are *Finslerian* [7,8] as their metric function is not limited by quadratic forms. Secondly they belong to the class known in the Finslerian geometry under the name of Minkowskian space [9], with which it is customary to associate the manifold where the indicatrices do not depend on the point. [The space of the Special theory of Relativity is a specific case of such spaces.] But the examined class of spaces is even smaller, as it is related to a strict idea of polylinear symmetric form. The latter case has a great significance as it becomes possible to introduce characteristics, that generalize such fundamental categories of geometry as the length, the angle, the orthogonality, the conformal reflection, etc. Let us conventionally call such spaces the *polylinear Finslerian spaces* (till the appearance of a more specific name let).

If $\mathbf{a}$ and $\mathbf{b}$, and also $\mathbf{a}'$ and $\mathbf{b}'$, are two pairs of unit vectors, then the figure, constructed with the first two vectors, will be congruent to the figure, constructed with the latter two, if a transformation mapping one figure onto the other there will be found. From the above examined properties of the polylinear forms it follows that such transformation can be found only if

$$\begin{gathered}(\mathbf{a}, \mathbf{a}, \ldots, \mathbf{b}) = (\mathbf{a}', \mathbf{a}', \ldots, \mathbf{b}'),\\ (\mathbf{a}, \mathbf{a}, \ldots, \mathbf{b}, \mathbf{b}) = (\mathbf{a}', \mathbf{a}', \ldots, \mathbf{b}', \mathbf{b}'),\\ \cdots\cdots\cdots\\ (\mathbf{a}, \mathbf{b}, \ldots, \mathbf{b}) = (\mathbf{a}', \mathbf{b}', \ldots, \mathbf{b}'). \qquad (2)\end{gathered}$$

This, in particular, entails that in the bilinear spaces the congruence of the pair of two unit vectors is related to the equality of only one form:

$$(\mathbf{a},\mathbf{b}) = (\mathbf{a}',\mathbf{b}'), \tag{3}$$

which sets the idea of the angle as the parameter that characterizes the difference between two directions. The equality (3) along with the definition of the unit vector are tantamount to the axiom of the triangle congruence from the Hilbert system of axioms of the Euclidean space. Two triangles are congruent in the Euclidean space if the lengths of corresponding sides and angles between them are equal. One may can formulate analogous axioms also for the pseudo-Euclidean spaces. But it follows from the definition (2) that in the polylinear space with the dimension of the form of more than two the congruence of figures constructed of two unit vectors is defined by more than one circumstance. In the spaces with the three-linear form $(\mathbf{a},\mathbf{b},\mathbf{c})$, the two forms must be equal to ensure that the figures would be congruent:

$$(\mathbf{a},\mathbf{a},\mathbf{b}) = (\mathbf{a}',\mathbf{a}',\mathbf{b}'), \qquad (\mathbf{a},\mathbf{b},\mathbf{b}) = (\mathbf{a}',\mathbf{b}',\mathbf{b}').$$

This seeming paradox has a very simple explanation. Usually speaking about a spatial figure, constructed on two vectors, it is thought as of a plain element held among sides, which are the driving vectors. But this is justified only in spaces with the bilinear form. In the spaces with the arbitrary polylinear form, the two vectors are now connected not with a plane but with a special cone-shaped surface, which configuration depends on the metric properties of the surrounding space. There can be more than one parameter, that defines the congruence of such fan-shaped figures, limited in the edges by unit vectors, that in particular is observed in spaces with three-linear symmetric form with two corresponding values.

On the basis of the above given brief analysis it becomes clear that polylinear spaces admit an introduction of analogous of the idea of the angle attributed to bilinear spaces. But we should take into account that the angle as the parameter in the bilinear spaces unifies simultaneously two properties: on the one hand, it serves as a characteristic of the difference between two directions, and on the other hand, is the parameter of one of types of congruent transformations called rotations. In the general case of the polylinear space each of the properties should be characterized by a proper value. It is meaningful to use the negative value of the n- polyform of the difference as the basis to getting the numerical parameter that would characterize the difference of directions of unit vectors: $\mathbf{a}$ and $\mathbf{b}$, to be more specific:

$$(\mathbf{a}-\mathbf{b},\mathbf{a}-\mathbf{b},\ldots,\mathbf{a}-\mathbf{b}) =$$
$$= (\mathbf{a},\mathbf{a},\ldots,\mathbf{a}) - C_n^1(\mathbf{a},\mathbf{a},\ldots,\mathbf{a},\mathbf{b}) \pm \ldots (-1)^{n-1}C_n^{n-1}(\mathbf{a},\mathbf{b},\ldots,\mathbf{b}) + (-1)^n(\mathbf{b},\mathbf{b},\ldots,\mathbf{b}),$$

where C_i^j are binomial coefficients. Consequently the scalar form of two unit vectors $\mathbf{a}$ and $\mathbf{b}$ reads

$$S(\mathbf{a},\mathbf{b}) = -C_n^1(\mathbf{a},\mathbf{a},\ldots,\mathbf{a},\mathbf{b}) \pm \ldots (-1)^{n-1}C_n^{n-1}(\mathbf{a},\mathbf{b},\ldots,\mathbf{b}) \tag{4}$$

or its function can play the role of a numerical parameter that defines the required property. Let us note that if the polylinear space is a two-bilinear one the expression (4) to the constant factor coincides with the definition of the common scalar product of two unit vectors. The value (4) can be called the *scalar product of two vectors* of the polylinear space. But may be it is even justified to divide the scalar product into items symmetrized in pairs:

$$S(\mathbf{a},\mathbf{b}) = C_n^1(-(\mathbf{a},\mathbf{a},\dots,\mathbf{a},\mathbf{b}) + (-1)^{n-1}(\mathbf{a},\mathbf{b},\mathbf{b}\dots,\mathbf{b}))$$
$$+ C_n^2((\mathbf{a},\mathbf{a},\dots,\mathbf{a},\mathbf{b},\mathbf{b}) + (-1)^{n-2}(\mathbf{a},\mathbf{a},\mathbf{b}\dots,\mathbf{b})) \pm \dots = S_1(\mathbf{a},\mathbf{b}) + S_2(\mathbf{a},\mathbf{b}) + \dots, \quad (5)$$

where every term $S_i(\mathbf{a},\mathbf{b})$ receives its proper value.

In the polylinear spaces there are pairs of vectors with definite ability of positional relationship similar to orthogonal vectors in the bilinear spaces. In the Finslerian space theory the corresponding idea is called the transversality. Let us call the vector $\mathbf{A}$ *transversal* to the vector $\mathbf{B}$, if $(\mathbf{A},\mathbf{A},\dots,\mathbf{A},\mathbf{B}) = 0$. It is seen here that the transversality is not commutative, that is, the vanishing $(\mathbf{A},\mathbf{A},\dots,\mathbf{A},\mathbf{B}) = 0$ does not entail $(\mathbf{B},\mathbf{B},\dots,\mathbf{B},\mathbf{A}) = 0$. But if we use the symmetrized forms (5), then the transversality, assigned by them, will have commutative properties. By definition, we will consider $\mathbf{A}$ and $\mathbf{B}$ *mutually transversal of the first degree*, when $S_1(\mathbf{A},\mathbf{B}) = \mathbf{0}$; and of the *second degree*, if $S_2(\mathbf{A},\mathbf{B}) = \mathbf{0}$, and so on up to $n/2$ or $(n-1)/2$ degree. Such differentiation of transversality demonstrates the ability of vectors of the linear Finslerian spaces to form pairs with a multitude of characteristic connection with the direction, – that generalizes the conception of orthogonality.

Apart from the quantities defined by the forms (4) it is meaningful to introduce one more "angle-like" characteristic in some polylinear spaces that have continuous congruent transformations like rotations. We will relate its value with the arc length in the unit sphere outlined by a ray simultaneously with a continuous one-parameter rotation. So generalized conception includes the property of the common angle - to be the additive measure that follows from the additivity of the length.

Not only pairs can be included into polyforms, but also three-, four-, etc., up to n different vectors. It is difficult to say to which quality consequences must lead this circumstance in the area of simple figures. Only one thing is clear: this property of polylinear spaces exists objectively that means that it should be as well taken into account.

There are such spaces among the polylinear ones where in one of the bases all the forms are nullified but for the ones that include only different vectors. For such spaces the fundamental metric forms take the following structure in the special basis:

$$(\mathbf{A},\mathbf{A},\dots,\mathbf{A}) = \pm a_1a_2\dots a_m \pm a_1a_2\dots a_{m-1}a_{m+1}$$
$$\pm \dots \pm a_2a_3\dots a_ma_{m+1} \pm \dots \pm a_{n-m}a_{n-m+1}\dots a_n. \quad (6)$$

Among these emerge the pseudo-Euclidean spaces labeled $(1, m-1)$, which play an important role in the modern theoretical physics. Though the classical quadratic form seems to be more convenient for the spaces, the second degree of the intervals in some of the isotropic bases looks like:

$$|\mathbf{A}|^2 = (\mathbf{A},\mathbf{A}) = a_1a_2 + a_1a_3 + a_1a_4 + \dots + a_{m-1}a_m = \sum_{k\neq l} a_ka_l.$$

For example, the square of interval of Minkowskian space $S^2 = (ct)^2 - x^2 - y^2 - z^2$ after the substitution

$$ct = \sqrt{3/8}(u+v+w+z), \quad x = \sqrt{1/8}(u-v+w-z),$$
$$y = \sqrt{1/8}(u+v-w-z), \quad z = \sqrt{1/8}(u-v-w+z)$$

(similar to (16)) gets an attractive symmetric form:

$$S^2 = uv + uw + uz + vw + vz + wz.$$

The expression (6) looks more concise in the cases with $n = m$, that is, when the dimension of the fundamental form coincides with the dimension of the space. In this case the nth degree of a vectors with respect to the corresponding basis takes on the form

$$|\mathbf{A}|^n = (\mathbf{A}, \mathbf{A}, \ldots, \mathbf{A}) = \pm a_1 a_2 \ldots a_n.$$

In these circumstances the specific role of the pseudo-Euclidean plane, where such correlations are held, is defined. It seems probable that there must exist a connection with associative-commutative algebras, that involves the appearance in the space of a large group of conformal reflections, only in spaces with $n = m$. At the same time the conformal reflections can be seen in a number of cases which follow from the works [10, 11] where the eight-dimensional biquaternions are examined, that, according to the above given axiom, have metric forms of the fourth degree which come outside the Liouville theorem. We can only hope that the property of some polylinear spaces has a vast group of conformal reflections which appears to be perspective in geometry as well as in physics.

On the other hand even superficial study of the properties of the polylinear spaces let us state that in some of them there are not only conformal, but also non-linear transformations that do not have analogies within common bilinear spaces. The presence of such transformations ensues merely from that the studied spaces require extension of the notion of orthogonality up to several respective members. As is well known, the nonlinear transformations that leave invariant ordinary orthogonality relates to conformal. In this connection it is natural to expect that the transformations retaining the transversality would occur preferable, too. This makes the existent polylinear spaces even more interesting.

4. Examples of polylinear spaces

There is a great number of polylinear spaces. The task to classify such spaces seems to be difficult even if we work with three-linear forms, not to mention the forms with a larger number of dimensions. But if we limit ourselves to the three-dimensional case, and if among symmetric three-dimensional spaces we examine those whose metric forms do not depend on permutation of vector components (it is suggested in the work [12], that examines a similar classification, to call them the *high-symmetric*) than we can single out 8 independent classes, where a fundamental canonical polyform can be related to each of them. The simplest look among all the forms has the following:

$$(\mathbf{A}, \mathbf{A}, \mathbf{A}) = a_1^3 + a_2^3 + a_3^3 = F_1;$$

$$(\mathbf{A}, \mathbf{A}, \mathbf{A}) = a_1^2 a_2 + a_1^2 a_3 + a_2^2 a_1 + a_2^2 a_3 + a_3^2 a_1 + a_3^2 a_2 = F_2;$$

$$(\mathbf{A}, \mathbf{A}, \mathbf{A}) = a_1 a_2 a_3 = F_3.$$

In the work [12] they are called *basic*. Any of the eight non-isomorphic high-symmetric tree-linear polyforms can be presented as a linear combination of the bases:

$$(\mathbf{A}, \mathbf{A}, \mathbf{A}) = \omega_1 F_1 + \omega_2 F_2 + \omega_3 F_3.$$

But no matter how great the variety of spaces with three-linear symmetric form is, the space with the following form stands out with its concise symmetry:

$$(\mathbf{A}, \mathbf{A}, \mathbf{A}) = a_1 a_2 a_3.$$

As the result of its high involved symmetry we can confront the corresponding space with the algebra of commutative-associative numbers that is the sum of three real algebras. Let us call such hyper-complex system the *triple numbers* and label it as H_3. Mathematical, geometrical and may be physical structures related to the triple numbers are not trivial at all, that is proved in the works [13, 14] published in this issue. It will be noted that most three-linear polyforms cannot be juxtaposed by algebras in general [12].

In the four-dimensional polylinear spaces with $n = m$ the basic forms have the following shape:

$$(\mathbf{A}, \mathbf{A}, \mathbf{A}, \mathbf{A}) = a_1^4 + a_2^4 + a_3^4 + a_4^4; \tag{7}$$

$$(\mathbf{A}, \mathbf{A}, \mathbf{A}, \mathbf{A}) = a_1^3(a_2 + a_3 + a_4) + a_2^3(a_1 + a_3 + a_4) \\ + a_3^3(a_1 + a_2 + a_4) + a_4^3(a_1 + a_2 + a_3); \tag{8}$$

$$(\mathbf{A}, \mathbf{A}, \mathbf{A}, \mathbf{A}) = a_1^2a_2^2 + a_1^2a_3^2 + a_1^2a_4^2 + a_2^2a_3^2 + a_2^2a_4^2 + a_3^2a_4^2; \tag{9}$$

$$(\mathbf{A}, \mathbf{A}, \mathbf{A}, \mathbf{A}) = a_1^2(a_2a_3 + a_2a_4 + a_3a_4) + a_2^2(a_1a_3 + a_1a_4 + a_3a_4) + \\ a_3^2(a_1a_2 + a_1a_4 + a_2a_4) + a_4^2(a_1a_2 + a_1a_3 + a_2a_3); \tag{10}$$

$$(\mathbf{A}, \mathbf{A}, \mathbf{A}, \mathbf{A}) = a_1a_2a_3a_4, \tag{11}$$

and to each of them their particular, not isomorphic to others, geometries of the polylinear space.

As well as in the three-dimensional case the variety of four-dimensional polylinear spaces is not limited to these examples. It seems to be a very difficult task to present the full classification of corresponding geometries. Let us study at least one case before setting about its realization. For example, the geometry related to the most symmetric among the basic polyforms (7) – (11), and to be more specific (11). Its high symmetry again gives us an opportunity to confront the space defined by it to the algebra of commutative-associative hyper-complex numbers, that in order to be brief we will call the *Quadra numbers* labeled as H_4. Some of the properties of the space, related to the Quadra numbers are given in [15]. We can get the Quadra number algebra by adding the axiom of real numbers to the axiom of composing and multiplication of the following objects: $A = a_1 \cdot 1 + a_2 \cdot I + a_3 \cdot J + a_4 \cdot K$ and $B = b_1 \cdot 1 + b_2 \cdot I + b_3 \cdot J + b_4 \cdot K$, where a_i and b_i – real numbers called the components, and $1, I, J, K$ the basic units. We accepting by definition that the sum of the numbers A and B is called the number

$$C = (a_1 + b_1) \cdot 1 + (a_2 + b_2) \cdot I + (a_3 + b_3) \cdot J + (a_4 + b_4) \cdot K,$$

and their product – another number of the same class:

$$D = (a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4) \cdot 1 + (a_1b_2 + a_2b_1 + a_3b_4 + a_4b_3) \cdot I + \\ + (a_1b_3 + a_2b_4 + a_3b_1 + a_4b_2) \cdot J + (a_1b_4 + a_2b_3 + a_3b_2 + a_4b_1) \cdot K,$$

By the above given method we get the algebra of commutative-associative hyper-complex numbers the, where the multiplication table of basic units have the following look:

	1	I	J	K
1	1	I	J	K
I	I	1	K	J
J	J	K	1	I
K	K	J	I	1

It follows from the table that $I^2 = J^2 = K^2 = 1$, namely all its imaginary units are hyperbolic. We can get the same algebra another way: by applying for 2 times the agebra of the real number using two independent hyperbolical-imaginary units I and J the doubling operation. Let us denote the product of I and J as an independent object k, the number A from the corresponding multitude can be presented as a linear combination:

$$A = (a_1 + a_2 \cdot I) + (a_3 + a_4 \cdot I) \cdot J = a_1 + a_2 \cdot I + a_3 \cdot J + a_4 \cdot K,$$

where the symbol of the real unit 1, as it is accepted in the complex-numbers and quaternions, is omitted.

Let us call the numbers $\bar{A}, \hat{A}, \widetilde{A}$ *conjugate* to the number $A = a_1 + a_2 \cdot I + a_3 \cdot J + a_4 \cdot K$, if they look like:

$$\begin{aligned} \bar{A} &= a_1 - a_2 \cdot I + a_3 \cdot J - a_4 \cdot K, \\ \hat{A} &= a_1 + a_2 \cdot I - a_3 \cdot J - a_4 \cdot K, \\ \widetilde{A} &= a_1 - a_2 \cdot I - a_3 \cdot J + a_4 \cdot K. \end{aligned} \tag{12}$$

Notice that

$$\hat{\bar{\widetilde{A}}} = A. \tag{13}$$

The product of such fours, as it is easy to check by the direct substitution, are always real numbers

$$A\bar{A}\hat{A}\widetilde{A} = a_1^4 + a_2^4 + a_3^4 + a_4^4 - 2a_1^2a_2^2 - 2a_1^2a_3^2 - 2a_1^2a_4^2 - 2a_2^2a_3^2 - 2a_2^2a_4^2 - 2a_3^2a_4^2 + 8a_1a_2a_3a_4. \tag{14}$$

By analogy with the algebra of complex numbers we will relate the value to the fourth degree of the corresponding number modulus and denote it as $|A|^4$. The introduced conception has the common properties of the modulus:

$$|\lambda A| = |\lambda| \cdot |A|, \qquad |AB| = |A| \cdot |B|,$$

where λ is a real, and A, B are complex numbers. In the product the property of mutually conjugated to result in the real number let us introduce into the examined algebra the operation of division, interpreted as an action inverse to multiplication. So, let us understand the number

$$A^{-1} = \frac{\bar{A}\hat{A}\widetilde{A}}{|A|^4} \tag{15}$$

under the number A^{-1} which is inverse to A. Only the numbers whose module is non-zero have their inverse analogues. Such numbers do not have such analogs. The examined algebra is associated with the form (11). It can be proved by examining a shift from the basis $1, I, J, K$ to the basis $S_1, S_2, S_3, S_4,$, whose objects are connected with the initial correlation:

$$\begin{aligned} S_1 &= \frac{1}{4}(1 + I + J + K), \qquad & S_2 &= \frac{1}{4}(1 - I + J - K), \\ S_3 &= \frac{1}{4}(1 + I - J - K), \qquad & S_4 &= \frac{1}{4}(1 - I - J + K). \end{aligned} \tag{16}$$

These bases are the divisors of zero and are distinguished by the fact that their multiplication table is the most vivid one:

	S_1	S_2	S_3	S_4
S_1	S_1	0	0	0
S_2	0	S_2	0	0
S_3	0	0	S_3	0
S_4	0	0	0	S_4

.

We will call the divisor of zero with such properties the *principle*, and the bases formed of them – the *absolute*. The feedback of the units $1, I, J, K$ with the principle zero divisor of the algebra H_4 is evaluated the following way:

$$1 = S_1 + S_2 + S_3 + S_4, \qquad I = S_1 - S_2 + S_3 - S_4,$$

$$J = S_1 + S_2 - S_3 - S_4, \qquad K = S_1 - S_2 - S_3 + S_4.$$

It is easy not only to sum but also multiply and divide the numbers from H_4 written in the absolute basis. So, the product of two numbers A and B looks is following:

$$(AB) = (a'_1 b'_1)S_1 + (a'_2 b'_2)S_2 + (a'_3 b'_3)S_3 + (a'_4 b'_4)S_4,$$

and their fraction reads

$$\frac{A}{B} = \frac{a'_1}{b'_1}S_1 + \frac{a'_2}{b'_2}S_2 + \frac{a'_3}{b'_3}S_3 + \frac{a'_4}{b'_4}S_4.$$

(Henceforth the components with primes will relate to the absolute basis). the absolute basis reveals the structure of the quadrahypeboloic number algebra, which is isomorphic to the algebra of real diagonal matrices. The group of mutually conjugated written in the absolute basis looks like:

$$\begin{aligned}
A &= a'_1 S_1 + a'_2 S_2 + a'_3 S_3 + a'_4 S_4,\\
\bar{A} &= a'_2 S_1 + a'_1 S_2 + a'_4 S_3 + a'_3 S_4,\\
\hat{A} &= a'_3 S_1 + a'_4 S_2 + a'_1 S_3 + a'_2 S_4,\\
\widetilde{A} &= a'_4 S_1 + a'_3 S_2 + a'_2 S_3 + a'_1 S_4.
\end{aligned} \tag{17}$$

The modulus of the number A in such special basis looks like:

$$|A| = |a'_1 a'_2 a'_3 a'_4|^{1/4}, \tag{18}$$

that proves the correspondence of the algebra to geometry defined by the fundamental metric form (11). We can introduce the conception of function for the multitude of the Quadra numbers. The exponential function is one of the most interesting. Under it we will understand the following series:

$$e^X = 1 + X + \frac{X}{2!} + \ldots,$$

where X is an arbitrary Quadra number. With the introduction of the exponential function we can examine along with the algebraic form of the number H_4 its exponential form. So, the number $A = a'_1 S_1 + a'_2 S_2 + a'_3 S_3 + a'_4 S_4$, where all the components of a'_i in the absolute basis are positive, corresponds to:

$$A = |A| e^{\alpha I + \beta J + \gamma K}, \tag{19}$$

where the positive value $|A|$ is its modulus. By analogy with the complex and double numbers we will call the real numbers α, β and γ, the *argument of the Quadra number A*. The connection of the arguments with the components a'_i in the absolute basis looks like:

$$\alpha = \frac{1}{4}\ln\frac{a'_1 a'_3}{a'_2 a'_4} = \frac{1}{4}(\ln a'_1 - \ln a'_2 + \ln a'_3 - \ln a'_4),$$

$$\beta = \frac{1}{4}\ln\frac{a_1'a_2'}{a_3'a_4'} = \frac{1}{4}(\ln a_1' + \ln a_2' - \ln a_3' - \ln a_4'),$$
$$\gamma = \frac{1}{4}\ln\frac{a_1'a_4'}{a_2'a_3'} = \frac{1}{4}(\ln a_1' - \ln a_2' - \ln a_3' + \ln a_4'),$$

where $\ln x$ is a logarithmic function of the real x. As the hyperboloic analog to the Euler formula works for every imaginary unit:

$$e^{\alpha I} = \cosh\alpha + I\sinh\alpha,$$

then the following expression for the exponent from an arbitrary Quadra number $X = \delta + \alpha I + \beta J + \gamma K$ is true:

$$e^X = (\cosh\delta + \sinh\delta)(\cosh\alpha + I\sinh\alpha)(\cosh\beta + J\sinh\beta)(\cosh\gamma + K\sinh\gamma), \quad (20)$$

where $\cosh x$ and $\sinh x$ are hyperbolic sinus and cosine. We can introduce an analogous function for the quadranumerical variable X as the following rows:

$$\cosh X = 1 + \frac{X^2}{2!} + \ldots, \qquad \sinh X = X + \frac{X^3}{3!} + \ldots .$$

We can connect the notion of the derivative with the function of the quadranumerical variable by the direction and analyticity the same way as the corresponding ideas are introduced into the algebra of double numbers [2]. The analyticity of the function from H_4 denotes the independence of its derivative from directions, [5] $dF = F'da$, and appears in simultaneous execution of 12 equations, which are analogs to the Cauchy-Riemann terms for the complex and double variables:

$$\frac{\partial U}{\partial a_1} = \frac{\partial V}{\partial a_2} = \frac{\partial W}{\partial a_3} = \frac{\partial Q}{\partial a_4}, \qquad \frac{\partial U}{\partial a_2} = \frac{\partial V}{\partial a_1} = \frac{\partial W}{\partial a_4} = \frac{\partial Q}{\partial a_3},$$
$$\frac{\partial U}{\partial a_3} = \frac{\partial V}{\partial a_4} = \frac{\partial W}{\partial a_1} = \frac{\partial Q}{\partial a_2}, \qquad \frac{\partial U}{\partial a_4} = \frac{\partial V}{\partial a_3} = \frac{\partial W}{\partial a_2} = \frac{\partial Q}{\partial a_1}, \quad (21)$$

where

$$F(A) = U(a_1, a_2, a_3, a_4) + V(a_1, a_2, a_3, a_4)I + W(a_1, a_2, a_3, a_4)J + Q(a_1, a_2, a_3, a_4)K$$

is an analytical function of a quadranumerical variable, and U, V, W, Q are hypercomplex-conjugated functions of four real arguments. In the algebra of quadranumbers there are 16 typical unit objects $e_1 - e_{16}$ that have in their basis, where the form (11) is written, the following components:

$$e_1 \leftrightarrow (1,1,1,1); \qquad e_5 \leftrightarrow (-1,-1,-1,-1);$$
$$e_2 \leftrightarrow (1,-1,1,-1); \qquad e_6 \leftrightarrow (-1,1,-1,1);$$
$$e_3 \leftrightarrow (1,1,-1,-1); \qquad e_7 \leftrightarrow (-1,-1,1,1);$$
$$e_4 \leftrightarrow (1,-1,-1,1); \qquad e_8 \leftrightarrow (-1,1,1,-1);$$
$$e_9 \leftrightarrow (1,-1,-1,-1); \qquad e_{13} \leftrightarrow (-1,1,1,1);$$
$$e_{10} \leftrightarrow (1,1,-1,1); \qquad e_{14} \leftrightarrow (-1,-1,1,-1);$$
$$e_{11} \leftrightarrow (1,-1,1,1); \qquad e_{15} \leftrightarrow (-1,1,-1,-1);$$
$$e_{12} \leftrightarrow (1,1,1,-1); \qquad e_{16} \leftrightarrow (-1,-1,-1,1).$$

The vectors e_i that correspond to the numbers can be used to illustrate the presence in the Quadra space of two types of transverslity, that generalize the idea of orthogonal directions for the Finslerian space. This is true that the 2 symmetrized forms (5) enter the Quadra space. They look like:

$$S_1(\mathbf{a}, \mathbf{b}) = (\mathbf{a}, \mathbf{a}, \mathbf{a}, \mathbf{b}) + (\mathbf{a}, \mathbf{b}, \mathbf{b}, \mathbf{b}) \tag{22}$$

and

$$S_2(\mathbf{a}, \mathbf{b}) = (\mathbf{a}, \mathbf{a}, \mathbf{b}, \mathbf{b}). \tag{23}$$

The equality to zero of any of them means the transversality of the corresponding directions. By direct substitutions of the components of vectors e_i in (22) and (23) we can make ourselves absolutely sure of the fact that every vector of the multitude faces 1, form mutually transversal pairs of the first order, and of the second with 8 of them. We can construct the basis that is an analog to the orthogonal from the four the first order transversal vectors. One of the specific cases of the basis is the above examined four-set $1, I, J, K$. It is impossible to construct basis from the second order transversal vectors as for each pair of the third and what is more fourth order do not have such correlation of directions.

5. Conclusion

The offered method of studying the examined class of Finslerian linear spaces, called polylinear, seems to be promising for it is based on the same principles as the scalar product. Let us note that the arising abilities let us move the focus of studies from the common vivid base to the soil of mathematical constructions. Thus the pseudo-Euclidean spaces demonstrate advantages of the analogous substitution. Not all geometrical effects are vivid in these spaces but the extension of the scalar product in its time was very useful.

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