ON SOME QUESTIONS OF FOUR DIMENSIONAL TOPOLOGY: A SURVEY OF MODERN RESEARCH

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Introduction

Our physical intuition distinguishes four dimensions in a natural correspondence with material reality. Four dimensionality plays special role in almost all modern physical theories. High dimensional quantum fields theory and string theory are considered together with their compactifications, i. e. the main space, describing the reality is a product of a four-dimensional manifold with some compact high-dimensional space. In this way we come to the well-known Kaluza-Klein model and ten-dimension superstring theory.

It is an interesting fact that the dimension four is a more complicated dimension from pure mathematical point of view. It seems that there is a contradiction with our intuition in understanding of the dimension concept, really, new dimensions give us new complexity. But it is not true in general. Additional dimensions often give a new freedom. It is natural that we must have some golden mean in this approach, in which we don't have a necessary freedom, but low-dimensional methods weakly work. In topology this mean is dimension four.

The goal of this note is to give a small survey of some problems in four-dimensional topology.

S-cobordism problem

One of the main questions of geometric topology is the problem to classify manifolds lying in a given category with respectively chosen equivalence relation. Working in the topological category, the question about classification of topological manifolds up to homeomorphism rises, for example, assuming compactness, connecteness and closedness. In dimension one we have only circles, in dimension two we come to the complete classification: every connected closed compact manifold is homeomorphic to the two-dimensional sphere with handles and Mobius bands. In this case, the fundamental group is a complete topological invariant. In the dimension three the question about classification becomes a hard problem, the existence of the connected 1-connected three-dimensional manifold, which is not homeomorphic to the three-dimensional sphere, is a well-known Poincare Problem. It is interesting that in dimension ≥ 5 many difficulties, occurring place in low dimensions, are disapear. First of all, this fact is connected with the concept of general position in high-dimensional spaces. Roughly speaking, in many important cases small deformations give possibility to cancel self-intersections of complexes. But in low-dimensional case we can not do the same.

Let's introduce one of the central equivalence relation in the topology of manifolds, so called s-cobordism relation. Let M_1 and M_2 be n-dimensional manifolds. We say that they are *cobordant* if there exists a $(n+1)$-dimensional manifold W, such that $\partial W = M_1 \cup M_2$. Further, if the embeddings $M_i \to W, i = 1, 2$ are homotopical equivalences, then this cobordism is called $h-cobordism$ (and manifolds are $h-cobordant$). Every homotopical equivalence defines an element from the Whitehead group, which depends only on the fundamental group of a given manifold (or in general, fundamental group of cell complex). The Whitehead group can be defined as a quotient of the K_1-functor of the integral group ring of the fundamental group by the natural action of group. In this way, the homotopical equivalence represents a trivial element of the Whitehead group if and only if it is homotopic to the composition of elementary cell extensions and collapsings, i.e. so-called simple homotopy equivalence. H-cobordism with simple homotopy equivalence is called *s-cobordism.* In particular, every homotopy equivalence between 1-connected manifolds is homotopic to the simple one.

The main result of the high-dimensional topology is following Theorem (see [1], [2]).

s-cobordism Theorem. Let $n \geq 5$. The connected h-cobordism W between n-dimensional manifolds M_1 and M_2 is homeomorphic to the direct product $W \equiv M_1 \times I$, if and only if this cobordism is an s-cobordism.

In particular, if we consider only 1-connected manifolds then arbitrary h-cobordism between them is a direct product. The higher-dimensional Poincare Conjecture then follows from this, i.e. every homotopical sphere is homeomorphic to the standard one in dimension ≥ 5. The proof of the s-cobordism Theorem fails in the case of dimension four and analogical statement presents an open problem:

Problem. Does the s-cobordism Theorem hold in the dimension 4?

The proof of the high-dimensional s-cobordism Theorem is based on the handlebody decomposition of the manifold W and reduction of a given manifold to the structure of the direct product of M_i with interval. The crucial point in this method is so-called Whitney trick. It gives a possibility to cancel the intersection points of the immersed submanifolds due to the embedding of a 2-dimensional disk (Whitney's disk), (see [1]). The main obstruction to extend the proof on the case of dimension four is the fact that Whitney trick does not work in dimension four. Actually, it is well-known that every 2-dimensional complex can be isotopically reduced to the embedded one in the 5-dimensional manifold. But in dimension four it is not true in general and we can consider the Whitney's disk only as immersed one. This easy fact destroys all prove of the s-cobordism theorem in the case of dimension 4.

To get over the difficulties related to the immersed Whitney disc, some new methods have been developed. The method given by A. Casson is most effective. The meaning of this method is to paste a self-intersections step by step by new immersed discs. This process can be extended infinitely long but the neighborhood of the final 2-complex is a handle, which is homotopically equivalent to the standard one. This idea was used by M. Freedman in the proof of the topological Poincare Conjecture in dimension four.

In general, as it was mentioned above, the s-cobordism problem in dimension 4 is still open. The analog of the s-cobordism Theorem was proved by M. Freedman and P. Teichner in 1996 in the class of 4-dimensional manifolds with fundamental groups of the subexponential growth (more precisely, of the growth $\leq 2^n$) [4].

False and exotic 4-dimensional manifolds

There is a natural question of comparison of given equivalence relations, i.e. homotopical equivalence, homeomorphisms, diffeomorphisms, in the class of manifolds of a fixed dimension. So, any two continously homeomorphic smooth manifolds are diffeomorphic in the dimension less than four. The situation in dimension four is much more complicated.

A manifold N is called a *false copy* of the manifold M if N is homotopically equivalent to M but not homeomorphic to M. N is called an *exotic copy* of M if N and M are homeomorphic, but not diffeomophic as manifolds.

The existence of the false and exotic spheres is connected with the topological and smooth versions of the Poincare Conjecture respectively. The smooth Poincare Conjecture is true in the dimensions less than four: there are no exotic three (and less) dimensional spheres. The analysis of the high-dimensional question leads to the beautiful theory of exotic spheres: there exist 28 7-dimensional manifolds, which are homeomorphic to the standard 7-dimensional sphere, but not diffeomorphic, due to the wonderful result of Milnor. The most intriging case is again dimension four. This is the only dimension, in which the existence of the exotic spheres is still open.

The situation with exotic copies of $\mathbb{R}^4$ is also very surprising. It is known that there does not exist any exotic $\mathbb{R}^n$ in dimension $n \neq 4$ and the analogical question was open for a long time in dimension four. In eighties due to the results of Freedman and Donaldson it was proved that there exist infinitely many smooth pair-wise nondiffeomorphic four-dimensional manifolds, such that each of them is homeomorphic to $\mathbb{R}^4$. The proof of this fact essentially used the methods of mathematical physics: instantons, Yang-Mills connections etc (see [5]). One of the main invariants of 1-connected four-dimensional manifolds is so-called intersection form, i.e. symmetric bilinear form, define on the second cohomologies of a given manifold. Classical Whitehead's theorem says that two given 1-connected oriented closed smooth four-dimensional manifolds are homotopically equivalent if and only if they have isomorphic intersection forms. In this connection, there is an actual question to classify all symmetric bilinear forms which can be realized as intersection form for some four-dimensional manifold. M. Freedman has shown that every symmetric bilinear form can be realized as an intersection form of some compact 1-connected four-dimensional manifolds and that there exist no more than two manifolds with given form. Donaldson classified all intersection forms of smooth manifolds and concluded from this the existence of the exotic structures on $\mathbb{R}^4$. The structure of exotic $\mathbb{R}^4$ is very complicated and takes important place in modern research. There are still many open questions related to such manifolds. In particular, does there exist any exotic $\mathbb{R}^4$ such that it can not be divided by properly embedded $\mathbb{R}^3$ onto two exotic pieces (Problem 4.43 (D), [6]).

The false four-dimensional manifolds construction requires an application of other techniques. As it was mentioned above, there are no false four-dimensional spheres (four-dimensional topological Poincare Conjecture). Very often the question about homeomorphicity of a given homotopic four-dimensional manifolds is very difficult. One of the first such type examples of four-dimensional manifolds is Cappell-Shaneson construction (see [2]): there exists a false projective $\mathbb{R}P^4$, which is homotopically equivalent but not diffeomorphic to $\mathbb{R}P^4$. This space is not PL-homeomorphic to $\mathbb{R}P^4$.

Finishing this section let's present more open problems in dimension four, related to exotic structures. The reader can find many classical and modern problems of this type the Kirby Problem List [6] (see also [7]).

Problem (4.77 [6]): An exotic smooth structure on $\mathbb{R}^4$ with $\mathbb{R}^1$ is diffeomorphic to $\mathbb{R}^5$. How can we usefully see the exotic $\mathbb{R}^4$ in $\mathbb{R}^5$?

Problem (4.86 [6]): Do all closed, smooth 4-manifolds have more than one smooth structure? (The generalization of the smooth 4-dimensional Poincare Conjecture).

Problem (4.87 [6]): Does every non-compact, smooth 4-manifold have an uncountable number of smoothings?

Schoenflies Conjecture

Consider one more problem, which has the solution in all dimensions besides four. This problem is about knotting in codimension equal to one. Recall that the embedding $f : M^m \to N^{n+m}$ is called *locally-flat* if the image of each point in N^{m+n} has neighborhood U such that the pair $(Im(f) \cap U, U)$ is homeomorphic (piecewise-linearly, in the case we work in this category) to the pare $(D^m \times D^n, D^m \times \{0\})$.

Conjecture Let $f : S^n \to S^{n+1}$ be a piece-linear locally-flat embedding. Then $S^{n+1} \setminus im(f)$ is 2-component and the closure of each of the components is a piece-linear n-dimensional ball.

Roughly speaking, this conjecture states that n-dimensional sphere can not knot in $(n+1)$-dimensional one. This conjecture turn out to be true in dimensions $n + 1 \neq 4$. But in the case of dimension four, again we can not apply the methods which we use in other dimensions.

Finishing this note, we want to mention one more time that there exist not so much fields in mathematics which use so different methods as four-dimensional topology. The problems of four-dimensional topology lead to the difficult questions of group theory. This is a theory of growth in groups, Andrews-Curtis-type problems, lower central series in groups etc. Also we can see many applications of high-dimensional methods in dimension four, for example, surgery exact sequences, methods of the link and knot theory. The dimension four is the unique dimension from the topological point of view, where we can find so many application of different techniques and which has so many open problems, the development of new tecniques of algebra and topology will be needed for their solution.

Acknowledgment

The author thanks Jitendra K. Bajpai for the help with translation.

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